FEB 23 1945
Mathematics Library

Mathematical Tables

Mathematics Library

and other

Aids to Computation

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

RAYMOND CLARE ARCHIBALD
DERRICK HENRY LEHMER

WITH THE COÖPERATION OF

LESLIE JOHN COMRIE
SOLOMON ACHILLOVICH JOFFE

I • Number 9 • January 1945

Published by

THE NATIONAL RESEARCH COUNCIL

NATIONAL RESEARCH COUNCIL

DIVISION OF PHYSICAL SCIENCES

COMMITTEE ON MATHEMATICAL TABLES AND OTHER AIDS TO COMPUTATION

*Professor R. C. ARCHIBALD, *chairman*, Brown University, Providence 12, Rhode Island (R.C.A.)

*Professor H. BATEMAN, California Institute of Technology, Pasadena 5, California (H.B.)

*Professor S. H. CALDWELL, Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts (S.H.C.)

*Doctor L. J. COMRIE, Scientific Computing Service, Ltd., 23 Bedford Square, London, W.C. 1, England (L.J.C.)

*Professor H. T. DAVIS, Department of Mathematics, Northwestern University, Evanston, Illinois (H.T.D.)

*Doctor W. J. ECKERT, Director Nautical Almanac, U. S. Naval Observatory, Washington 25 (W.J.E.)

*Mister J. S. ELSTON, The Travelers, Hartford, Connecticut (J.S.E.)

*Professor D. H. LEHMER, 942 Hilldale Avenue, Berkeley 8, California (D.H.L.)

*Professor S. S. WILKS, Department of Mathematics, Princeton University, Princeton, New Jersey (S.S.W.)

Professor W. G. COCHRAN, Iowa State College of Agriculture and Mechanic Arts, Ames, Iowa

Professor A. T. CRAIG, State University of Iowa, Iowa City, Iowa

Professor C. EISENHART, Agronomy Building, University of Wisconsin, Madison, Wisconsin

Professor J. D. ELDER, Department of Mathematics, University of Michigan, Ann Arbor, Michigan

Doctor L. GOLDBERG, McMath-Hulbert Observatory, Route 4, Pontiac, Michigan

Professor P. W. KETCHUM, Department of Mathematics, University of Illinois, Urbana, Illinois

Miss C. M. KRAMPE, U. S. Naval Observatory, Washington

Professor T. KUBOTA, Tôhoku Imperial University, Sendai, Japan, Representative of the National Research Council of Japan

Professor A. N. LOWAN, 312 Schenectady Avenue, Brooklyn 13, New York

Doctor J. C. P. MILLER, (Department of Applied Mathematics, University of Liverpool), 18 Garthdale Road, Liverpool 18, England

Doctor W. A. SHEWHART, Bell Telephone Laboratories, 463 West Street, New York City

Doctor G. R. STIBITZ, Bell Telephone Laboratories, 463 West Street, New York City

Mister J. S. THOMPSON, Mutual Benefit Life Insurance Company, Newark, New Jersey

Professor I. A. TRAVIS, [Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, Pennsylvania] Lieutenant Commander Travis, Bureau of Ordnance, Navy Department, Washington, D. C.

Mister W. R. WILLIAMSON, Federal Security Agency, Social Security Board, Washington

* Member of the Executive Committee.

Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

Subscription price for each *calendar* year $3.00, payable in advance; ordinary single numbers each $1.00 (special number 7, *Guide to Tables of Bessel Functions*, $1.75), payable to the Division of Physical Sciences, National Research Council, 2101 Constitution Avenue, Washington, D. C. No reductions are made to Libraries or to Booksellers.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

Machines for Solving Algebraic Equations

1. Introduction. The search for mechanical means of solving algebraic equations has interested mathematicians for well over a century. Two early papers date back to the eighteenth century. Perusing a paper of 1758 by SEGNER,[1] in which the author proposes a universal method of discovering real roots of equations, based on what we should now call drawing the graph of the function $y = \sum a_i x^i$, ROWNING[2] in 1770 considered the possibility of drawing the graph of a polynomial continuously by local motion. Theoretically at least, a number of rulers could be linked together so that the pencil point on the last ruler would trace the required curve. But mechanical limitations of the day caused a reviewer to remark that "as this is a matter of curiosity rather than any use, . . . it is unnecessary to enter any further into it at this time." Theoretical methods developed since that day have depended for their usefulness on the degree of precision in the mechanisms constructed to carry theory into practice, a precision which has greatly increased in modern times.

The early mechanical equation-solvers were restricted to finding the real roots of equations with real coefficients. But certain electrical methods, starting with the one described by LUCAS in 1888, were able to handle complex roots, and even complex coefficients. The modern isograph is an electromechanical device for finding real or complex roots of algebraic equations. In addition to the machines for solving algebraic equations in a single unknown, other similar devices have been invented for the solution of simultaneous linear equations in several unknowns.

Two excellent surveys of earlier mechanisms appeared at the beginning of this century, one by MEHMKE[3] in 1902, revised by D'OCAGNE[3] in 1909, and the other by MORITZ[4] in 1905. A few years later GHERSI,[5] in his book of mathematical curiosities, included an illustrated account of some of the previously discovered hydrostatic and electric solvers of algebraic equations. A comprehensive survey of various dynamical methods of solving algebraic equations was given by RIEBESELL[6] in 1914. The summaries and bibliographies published in these papers have been very helpful in the preparation of this article, and will be made use of below without further acknowledgment.

The diverse methods which have been proposed for solving algebraic equations mechanically, other than the strictly numerical methods based upon the use of calculating machines, fall naturally into about six types, and we shall discuss these in the succeeding paragraphs, as follows: (2) Graphic and visual methods. (3) Kinematic linkages. (4) Dynamic balances. (5) Hydrostatic balances. (6) Electric and electromagnetic methods. (7) Methods of harmonic analysis. Of these the first four are usually restricted to real roots, whereas the last two may be used to find the complex roots of equations. All these types include machines both for algebraic equations in one unknown, and for simultaneous linear equations in several unknowns.

In our description of various devices, it will be less confusing to the reader if in most cases we adopt a standard notation for a polynomial whose zeros are to be found, which may differ in several instances from those used by the authors we quote. Let it be required to determine the roots $z = x + iy$,

($i^2 = -1$), of the algebraic equation

$$f(z) \equiv c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n = 0, \tag{1.1}$$

where the coefficients $c_m = a_m + ib_m$ may be real or complex. The letter R will denote any convenient upper bound for the absolute values of the roots of $f(z)$. A suitably chosen integral of $f(z)$ will be denoted by $F(z)$, and its zeros by $Z_1, \cdots, Z_{n+1}$. To denote real coefficients we shall write a_m instead of c_m. If only real roots are to be found, the variable z will be called x. Thus the notation

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0, \tag{1.2}$$

will imply the problem of finding real roots of a polynomial equation with real coefficients. In such cases y will often be used to denote $f(x)$.

2. Graphic and visual methods. Twenty-five years after Rowning's paper, LAGRANGE[7] described a graphic method of solving algebraic equations. To solve the equation $f(x) = 0$, Lagrange lays off on the y axis (L_0) the $n + 1$ directed segments $OB_0 = a_0$, $B_0B_1 = a_1$, $B_1B_2 = a_2$, $\cdots$, $B_{n-1}B_n = a_n$. The coordinates of the point B_m are seen to be $(0, b_m)$, if $b_m = a_0 + a_1 + \cdots + a_m$. A horizontal line through B_n intersects the vertical $L_1(x = 1)$ at the point $C_n(1, b_n)$. The line $B_{n-1}C_n$, with slope a_n, meets a suitably selected vertical line L_x in a point $P_{n-1}(x, b_{n-1} + a_n x)$; a horizontal through P_{n-1} meets L_1 in $C_{n-1}(1, b_{n-1} + a_n x)$; the line $B_{n-2}C_{n-1}$ with slope $a_{n-1} + a_n x$ meets L_x in $P_{n-2}(x, b_{n-2} + a_{n-1}x + a_n x^2)$. Successive points P_m are constructed in this way on L_x until finally the point P_0 is found, whose coordinates are $(x, f(x))$. The locus of P_0, for various lines L_x, is the graph of the polynomial $y = f(x)$, and the roots are found whenever P_0 lies on the x-axis.

Fewer construction lines are involved in the graphic method of LILL[8] (1867). If we define the algebraic quantities y_m by the successive relations

$$y_n = 0, \qquad y_{m-1} = -x(a_m - y_m), \qquad m = n, n-1, \cdots, 2, 1, \tag{2.1}$$

then

$$y_{n-1} = -xa_n, \qquad y_{n-2} = -x(a_{n-1} + xa_n), \cdots, y_0 = a_0 - f(x). \tag{2.2}$$

The problem is reduced to constructing successively the segments y_{m-1}, $m = n, n-1, \cdots, 1$ and finding x by trial so that $y_0 = a_0$. A rectangular framework introduces the coefficients as follows. Starting at O, lay off $OA_n = a_n$ as a directed segment along the x-axis, lay off $A_nA_{n-1} = a_{n-1}$ as a directed segment parallel to the y-axis, lay off $A_{n-1}A_{n-2} = a_{n-2}$ as a directed segment with the positive sense opposite to that of the x-axis; and continue at each stage to rotate the positive sense through 90°. Now for any assumed value of x draw a line through O with slope $-x$ intersecting A_nA_{n-1} (extended if necessary) at a point P_{n-1} (this is such that A_nP_{n-1} will equal y_{n-1}); draw a perpendicular to this line at P_{n-1} intersecting $A_{n-1}A_{n-2}$ in P_{n-2}; draw a perpendicular to $P_{n-1}P_{n-2}$ at P_{n-2}, etc., until finally a point P_0 is located on A_1A_0, such that $P_0A_0 = f(x)$. If x is so chosen that P_0 coincides with A_0, it will be a root of $f(x) = 0$.

This method of Lill was somewhat modified by CREMONA[9] in 1874, and was reviewed by Moritz.[4]

A graphic method published by CUNYNGHAME[10] in 1886 gives the real roots of an equation of the form $x^n + Ax + B = 0$ to two decimals, if the curve $y = -x^n$ be first drawn on a suitably large scale. The author's notation is cumbersome, but his idea can be expressed simply by the use of coordinates. Let a line of slope A be drawn through $(0, B)$. Then the intersections of the curve $y = -x^n$ with this line $y = Ax + B$ define the real roots of the given equation. In the case of the cubic, the author discusses the computation of the "impossible" (what we should call imaginary) roots. Let $P(x, y)$ be a real intersection of the line $y = Ax + B$ with the cubic $y = -x^3$, and let m be the slope of the line drawn from P, tangent to the cubic at another point. Then it is easily shown that the three roots of the equation are x and $-(x/2) \pm \sqrt{m - A}$. Hence if $m < A$, two roots are imaginary, but these are readily computed from the two slopes. The author suggests the use of a special protractor on which slopes can be read directly, and marks the abscissas directly on the curve C instead of along the x-axis to simplify reading of the roots.

Extending the graphic idea from the trinomial to numerical equations of four or five terms, MEHMKE[11] describes an apparatus which was displayed in a mathematical exhibit in Munich in 1893. The theory is based on the fact that if four curved scales in space are cut by a plane, then the four readings will satisfy a functional relation. In Mehmke's model, three of the scales are uniform vertical scales A, B, C, parallel to each other but not coplanar. A line is determined by a string fastened between a point marked u on the A scale and a point marked w on the C scale. A plane is then determined by viewing this line through a sliding eyepiece set at v on the B scale. For an equation of four terms, a non-uniform curved scale, suitably graduated, is viewed through the eyepiece and seen to cut this plane in one or more points which define the required roots. For an equation of five terms a one-parameter family of such curves is used. The curves are constructed as follows. Let the given equation be

$$f(x) = x^m + ux^n + vx^p + wx^q = f. \tag{2.3}$$

On a particular curve corresponding to the parameter f, the point marked x is such that its projection on a horizontal plane cutting the A, B, C scales in a triangle ABC would be the centroid of masses x^n, x^p, and x^q placed at A, B, C, respectively; and its vertical projection s on one of the three parallel scales A or B or C is made to be $s = (f - x^m)/(x^n + x^p + x^q)$. Hence the intersection of this curve with the visual plane locates the root of the equation

$$s = \frac{ux^n + vx^p + wx^q}{x^n + x^p + x^q} = \frac{f - x^m}{x^n + x^p + x^q}. \tag{2.4}$$

3. Kinematic linkages. The mathematical theory of kinematic linkages, discussed long before by Rowning, awaited the day of precision machinery before it could be considered practical. In the meantime various theoretical devices were discussed in mathematical papers. The equiangular linkage described by KEMPE[12] in 1873 is a device for obtaining real roots of equations with real coefficients. It is constructed of $n + 1$ links $L_0, L_1, \cdots, L_n$, of lengths $l_0, l_1, \cdots, l_n$, joined consecutively at points $P_0, P_1, \cdots, P_{n-1}$, so that

the first link OP_0 lies along the positive x-axis with its left end O at the origin, and so that each of the consecutive links is constrained to make the same exterior angle ϕ with its predecessor. The horizontal projection of the directed link $L_m(= \overrightarrow{P_{m-1}P_m})$ is then equal to $l_m \cos m\phi$, and hence the horizontal projection of the directed line $\overrightarrow{OP_n}$ joining the two end-points of the linkage is

$$l_0 + l_1 \cos \phi + l_2 \cos 2\phi + \cdots + l_n \cos n\phi. \tag{3.1}$$

To find the roots of an algebraic equation $f(x) = 0$ it is necessary first to rewrite the equation in the form (3.1) and then to find those angles ϕ_i for which the free end P_n of the linkage lies on the y-axis. If R is any convenient upper bound of the roots of $f(x)$, we may set $x = R \cos \phi$, and then calculate the Fourier coefficients of the trigonometric polynomial $f(R \cos \phi)$. The lengths l_m can be taken as any convenient multiples of these coefficients, which are linear combinations of the given coefficients a_m of $f(x)$.

Although Kempe did not attempt a description of a mechanical method for producing his equiangular linkage, such an attempt was made later by BLAKESLEY. In 1907 he published a paper entitled "Logarithmic lazytongs and lattice-works,"[13] and then in 1912 he used the lazytongs idea to devise a mechanical construction for Kempe's equiangular linkage.[14] The construction which he gave was erroneous, however, and did not really produce an appropriate linkage except in special cases. The following is Blakesley's construction. On opposite sides of each of the segments $P_{m-1}P_m$ of Kempe's open polygon $OP_0P_1 \cdots P_n$, two isosceles triangles $P_{m-1}A_mP_m$ and $P_{m-1}B_mP_m$ are drawn forming a kite-shaped quadrilateral $P_{m-1}A_mP_mB_m$. The points A_0 and B_0 are arbitrary points on the perpendicular bisector of OP_0, but the other vertices A_i and B_i are chosen successively as the points where $A_{i-1}P_{i-1}$ and $B_{i-1}P_{i-1}$, produced, meet the perpendicular bisector of $P_{i-1}P_i$. Then $2n - 2$ rods,—one through $A_{i-1}P_{i-1}A_i$ and one through $B_{i-1}P_{i-1}B_i$, $(i = 1, 2, \cdots, n - 1)$,—are hinged to each other in pairs at P_{i-1} and are further hinged at their common extremities A_i and B_i. Two rods connect A_0 and B_0 to O, and two rods connect A_n and B_n to P_n, but the vertices of Kempe's polygon are *not* connected rigidly to each other by rods $P_{i-1}P_i$. The joints P_i in this configuration can be shown to form an equiangular linkage as the lattice is articulated. Blakesley's error arose in asserting that the ratios of the segments OP_0, P_0P_1, P_1P_2, $\cdots$, etc., remain constant. If this were true we should indeed have a machine for applying Kempe's theory. But in fact, only the ratios of alternate segments remain constant in the general case, although all ratios happen to do so in such special cases as the logarithmic lazytongs in which the successive lengths form a geometric progression. Blakesley's device fails as an exact solver of algebraic equations.

It may be of interest nevertheless to note Blakesley's alternate method of transforming the given equation $f(x) = 0$ into a Fourier series. This he does by setting $x = \tan \phi$, and computing the Fourier series for the trigonometric polynomial $(\cos^n \phi) \cdot f(\tan \phi)$. Separate linkages represent the cosine and sine terms of the series, and their extremities are fastened at right angles to each other to produce a single kinematic linkage.

An entirely different mechanism for solving equations of the n^{th} degree, based essentially on the graphic method similar to that of Lagrange (1795),

was published in the same year (1912) by MUIRHEAD.[15] First, by a transformation of the polynomial which replaces a negative root by a positive root, or a root <1 by its reciprocal, the problem is reduced to finding a root >1. Two parallel shafts s and s' are separated by the variable distance $x-1$. In the same plane, on a line PP' perpendicular to s and s' (which cuts s in P and s' in P', and which we shall call the axis), two centers O and O' are fixed so that $OP = P'O' = 1$, $OP' = PO' = x$. To find a root >1 of the equation (1.2), mark a directed distance $PP_n = a_n$ on the shaft s and fix it mechanically, let a rod OP_n intersect s' in P'_n, lay off and fix mechanically the directed distance $P'_nP'_{n-1} = a_{n-1}$ on the shaft s', let a rod $O'P'_{n-1}$ meet s in P_{n-1}, and lay off $P_{n-1}P_{n-2} = a_{n-2}$ on s, etc. To avoid mechanical interference, the successive rods can be placed in parallel planes instead of in the same plane. The machine is articulated by pulling the shafts s and s' apart. Whenever the point P_0 or P'_0 lies on the axis, the corresponding root x can be read off.

In the same paper Muirhead described a similar mechanism for solving a set of simultaneous linear equations. Each of the variables x, y, z, etc., requires a pair of shafts (X, X', and Y, Y', and Z, Z' etc.), separated by a unit distance, whose distances from a fixed reference shaft are constrained to be x and $x+1$, y and $y+1$, z and $z+1$, etc. Each equation ($a_ix+b_iy+c_iz+d_i=0$, for example) requires a set of rods of which consecutive pairs are hinged on the X, Y, Z shafts, and are forcibly separated by directed distances a_i, b_i, c_i along the X, Y, Z shafts. The first rod is fixed at O on the reference shaft, and the last is separated by the distance d_i from O on the reference shaft. Ways are devised to avoid mechanical interference.

Two years earlier (in 1910) NÄBAUER[16] described a rather different kinematic machine for solving simultaneous linear equations. The fundamental unit in his machine consists of a horizontal rod bearing two equal gears of radius r in vertical planes, one of which rests and rolls on a horizontal turntable, and can be slid back and forth on its axis so as to rest at a required distance ar from the center of the turntable. As the turntable turns through an angle x, the two equal gears turn through an angle ax. To avoid slipping, nine gear circles, for $a = 1, 2, 3, \cdots, 9$, are provided on the turntable. A separate turntable mechanism is provided for each of the variables x, y, z, and the contributions ax, by, cz are added by differential gears to produce a given sum l. Three such sums $a_ix + b_iy + c_iz = l_i$ can be formed with three sets of turntables, mounted so that each of the variables x, y, z has a vertical axis for itself. If the sums l_1, l_2, l_3 are prescribed, then the variables x, y, z are determined. Since friction alone is insufficient to produce the desired constraint, gears are used. Since the gears restrict the coefficients a_i, b_i, c_i to one-digit integers, a method of successive approximation is outlined which overcomes this difficulty. To a person familiar with early models of the differential analyzer, this machine demonstrates some of the same ideas and the same difficulties.

Two decades later a large machine for the mechanical solution of simultaneous equations was built at the Massachusetts Institute of Technology. An interesting description of the machine by WILBUR[17] is accompanied by plates showing the machine and some of the details of its operation. Ten plates P_j, one for each of ten homogeneous variables, are mounted to rotate about parallel horizontal axes fastened in a rigid frame. Each plate contains

ten slots perpendicular to its axis, nine of which are provided with midrunners on which a positive or negative coefficient a_{ij} may be set approximately by hand, and then adjusted with micrometer screws to .0005 in. or 4 significant figures. The front slot on each plate is used for an accurate reading of $\sin \theta_j$, the sine of the angle of rotation of the plate from the horizontal. The nine slots in a given plate are each associated with one of nine given linear equations—which can be considered as homogeneous in ten unknowns to be read as $\sin \theta_j$, or as non-homogeneous in nine ratios of these unknowns. The ten midrunners corresponding to a single equation lie in a vertical plane and are connected by a 60 ft. flexible steel tape. Fastened at a point A, the tape rides on 10 ball-bearing top runners, from each of which it loops down to a pulley on a midrunner. Then it passes down through a clamp C to the bottom, where it passes under a set of 10 bottom runners, from each of which it loops up to a second pulley on the midrunner, and is finally made fast at its end B. When the clamp C is released, all the plates are free to rotate arbitrarily, since any decrease in length of an upper loop is compensated by an increase in the corresponding lower loop, and vice versa. The total lengthening of CB, measured along the tape, is

$$2\sum a_{ij} \sin \theta_j + 2D_i \sin \theta_0 \tag{3.2}$$

if the "coefficients" a_{ij} are set as displacements on nine of the plates P_j, and the "constants" D_i are set as displacements on the tenth plate P_0. To solve the equations

$$\sum a_{ij}x_j + D_i = 0 \tag{3.3}$$

the clamp C is made fast and one of the plates is rotated with an oscillating motion which constrains the others to move with it. Then the unknowns $x_j = (\sin \theta_j)/\sin \theta_0$ may be read from the machine.

Having found x_i (usually to within 1% of the largest unknown, but this depends upon the stability of the system), the left-hand sides of (3.3) are computed accurately to as many figures as desired, with a computing machine. If these values d_i do not approximate 0 sufficiently, they may be reset on the "constant" plate P_0 as new constants in a set of equations with the same coefficients a_{ij}, of which the solutions are the errors of the first approximations. By repeating this procedure, as high a degree of accuracy may be obtained as is desired.

4. Dynamic balances. A paper by BÉRARD[18] in 1810 was the first to suggest using an ordinary balance for finding real roots of algebraic equations. His idea was improved by LALANNE[19] in 1840. The principle of the Lalanne balance was to have weights, proportional to the coefficients $|a_m|$, exert forces placed at directed distances from a fixed reference line of $+x^m$ if a_m is positive, or of $-x^m$ if a_m is negative, so as to produce moments $a_m x^m$ whose sum should be in equilibrium. For convenience in staying within a finite portion of the plane, a weight $10|a_2|$ at distance $\pm x^2/10$ can replace a weight $|a_2|$ at distance $\pm x^2$, etc. Balanced on a horizontal knife-edge is a horizontal rectangle in which guide curves, representing $\pm x^m$ divided by a suitable power of 10, are marked either by rigid curved wires or by slots in a sheet of metal. From the guide curves hang appro-

priate weights representing the coefficients. The constant term c_0 is represented by a weight $|c_0|$ hung on one of two fixed hooks on opposite sides of the knife-edge. By means of a horizontal indicator slide, perpendicular to the knife-edge, all the hooks are constrained to hang at the same distance x from an initial reference line. As the slide is moved from left to right, the real roots are read off at positions of equilibrium.

COLLIGNON,[20] in 1873, extended Lalanne's balance in a theoretical way to a complex variable, hanging weights at points representing the complex numbers 1, z, z^2, etc.; but he was not successful in devising a practical machine to move the hooks continuously in the prescribed manner. In 1881 EXNER[21] devised a turning balance in which the weights are placed on spiral curves.

Systems of levers were hinged together in a machine designed by BOYS[22] in 1886. In this machine a set of $n+1$ horizontal axes are placed at levels $0, 1, 2, \cdots n$, above a fixed pivot. The even-numbered axes are equally spaced in a fixed vertical plane s and the odd-numbered ones are equally spaced at intermediate levels in a parallel vertical plane s' at distance $x+1$ from s. Intersecting the m^{th} axis at right angles there is a horizontal lever with weight pans at distances marked $+1$ and -1, of which the one whose sign is that of a_m is to carry a weight $|a_m|$ so as to produce a moment a_m. Positive moments are clockwise on the left-hand even axes and counterclockwise on the right-hand odd axes. The $m-1^{\text{th}}$ lever is supported by a sliding joint fastened to the positive pan on the m^{th} lever which is at distance x from the $m-1^{\text{th}}$ axis. Thus the weight $|a_n|$ on the n^{th} lever produces a moment $a_n x$ on the $n-1^{\text{th}}$ axis, to which is added the moment a_{n-1} from the scale pan. A total moment of $(a_{n-1}+a_n x)x$ is communicated to the $n-2^{\text{th}}$ lever, to which is added the moment a_{n-2} of the weight $|a_{n-2}|$, etc. Finally, on the bottom 0^{th} lever the total moment is $f(x)$. The two vertical planes s and s' are to be moved apart until equilibrium is reached. Real positive roots can thus be read off within limits determined by the size of the machine.

Ten years later (1896) GRANT[23] described a similar machine, with which roots from 1 to ∞ could be read off on a reciprocal scale, on which the number x is marked at a distance $1/x$ from the mark ∞. Instead of alternating the $n+1$ horizontal axes in two banks, as Boys did, Grant makes them all lie in an inclined plane, displacing them horizontally by worm gears from an initial vertical reference plane through the 0-axis in such a way that the m-axis is displaced m times as far from reference plane as the 1-axis is displaced. Scale pans are again hung at distances ± 1 from the axes, but the supports come at distances $1/x$ from the axes. This machine, like that of Boys, is used only to find the real roots, which are read at equilibrium positions. It is interesting, in view of later inventions, to note Grant's statement that "an imaginary root is as impossible mechanically as it is arithmetically."

Although a paper published by SKUTSCH[24] in 1902 refers primarily to hydrostatic balances, and will be discussed later in that connection, it should be mentioned here for its description of a kinematic circular balance. Like spokes of a wheel, a set of $n+1$ rods of length r emanate from a center O to points $A_0, A_1 \cdots A_n$; the whole figure being in a vertical plane. At each endpoint A_m is a geared pulley wheel (also in a vertical plane) around which is hung a chain suspending a weight pan with a weight adjusted to produce

a moment a_m on the m^{th} gear. A rod C_mD_m of length $2l$ is fastened rigidly to this m^{th} gear at its midpoint A_m. Connecting adjacent points C_m and D_{m+1} is a link C_mD_{m+1}, making an angle γ with C_mD_m and an angle δ with $D_{m+1}C_{m+1}$. The ratio $x = \sin\gamma/\sin\delta$ is adjusted to be the same for each of the n links. Then a balance of moments is obtained when x is a root of $f(x) = 0$.

Still another dynamic root-finder was described by PEDDIE[25] in 1912. A set of $n + 1$ pulleys, numbered from 0 to n, are so arranged in a vertical plane that the n^{th} pulley is fixed, the $n - 1^{\text{th}}$ is supported at its center by a string passing over the n^{th} pulley, and each successive pulley is supported by a string passing over the previous one, until the string over the last pulley (numbered 0) carries a weight to pull the strings all taut. The other ends of the $n + 1$ strings are unwound from drums and are all kept parallel at an inclination of θ with the horizontal, where $x = 1 + \sin\theta$ is the root to be found. If a length of string k is unwound from the pulley numbered m, then it can be shown that the pulley numbered $m - 1$ will unwind a length of string kx, and that this will finally cause a length kx^m to unwind from the bottom pulley (numbered 0). Hence if the machine is set by unwinding lengths of string equal to the coefficients $a_n, a_{n-1}, \cdots, a_0$ from the drum ends of the respective strings, the total displacement of the weight will be $f(x)$. At angles θ for which this displacement is 0, the root is read as $x = 1 + \sin\theta$. The method is obviously limited to finding real roots between 0 and 2, but can be used for finding the real roots of any polynomial if suitable preliminary transformations are first applied to the polynomial. A modification of this principle uses springs instead of gravity to pull the pulleys into position, and the pulleys are all placed on a movable arm which can turn from $\theta = 0°$ to $\theta = 90°$.

For further details on various dynamic balances, the reader is referred to Riebesell's paper.[6]

5. Hydrostatic balances. Closely related to the dynamic balances just discussed are the hydrostatic balances in which upward forces can be introduced which are proportional to the volumes displaced by appropriately chosen solids. A description by Ghersi of some of these methods has already been referred to above.[5]

The method proposed by DEMANET[26] in 1898 is adapted to the solution of the cubic equations $x^3 + x = c > 0$ and $x^3 - x = c > 0$. A cylindrical vessel and a conical vessel, both open at the top, are connected underneath by a pipe so that water in the two vessels will reach the same level. If the tangent of the semi-vertical angle of the cone is $\sqrt{3/\pi}$, then the volume of liquid in the cone will be x^3 when the surface is x units above the vertex. If the cylinder has unit cross-section, it will then contain x cubic units of water. Hence if c cubic units of water be added above the level of the vertex of the cone, the depth x will be the required root of $x^3 + x = c$. For the equation $x^3 - x = c$, a cylinder is used to displace a volume x of liquid within the cone. The cubic $y^3 \pm py = q$ can be solved by first writing $y = x\sqrt{p}$.

MESLIN'S[27] contribution in 1900 was a horizontal dynamic balance. Solids whose volumes, cut off at a level x, are proportional to $x, x^2, x^3, \cdots$, are hung on a lever at directed distances a_1, a_2, a_3 from the fulcrum, and the

constant a_0 is represented by a weight $|a_0|$ in one of two pans fastened on opposite ends of the lever. Before adding the weight $|a_0|$, the weights of the various solids have to be balanced by a suitable weight in a scale pan. Then the solids are suspended in two interconnected vessels, the weight $|a_0|$ is added to the proper scale pan, and water is made to flow from a third vessel until the balance is again restored to equilibrium. The depth x to which the solids are submerged gives the root. In his paper Meslin discusses in some detail the exact dimensions of the different solids to be used in his balance.

Two hydraulic methods were proposed by EMCH[28] in 1901 for extracting the n^{th} root of any number. The first method, which was extended in a second paper to the solution of an arbitrary equation of the n^{th} degree, was essentially the same as that of Meslin, and involved the depth of immersion of certain solids in water. The other, less accurate, depended upon the time required to empty a suitable vessel (whose interior is a surface of revolution), through a small orifice of area a in the bottom.

The radius r at height x is obtained as follows:

$$t = \sqrt[n]{x}, \qquad dt = \pi r^2 dx/a\sqrt{2gx} = (\sqrt[n]{x}/nx)dx, \tag{5.1}$$

or

$$r^2 = (a\sqrt{2g}/n\pi)\sqrt[n]{x}/\sqrt{x}. \tag{5.2}$$

A new approach to the same problem was made in 1902 by SKUTSCH[24] who considered not only horizontal positions of equilibrium on a hydrostatic balance, but also positions of equilibrium at an angle $\sin^{-1}\eta$ with the horizontal. After defining a balance as "a kinematic chain set up for the application of arbitrary forces, and used for finding relationships between the magnitude and position of these forces in equilibrium," Skutsch describes balances of MASSAU[29] (1878), Grant, and Meslin. He then uses a construction, similar to that of Meslin, but instead of the single depth variable x, he uses two variables ξ, η, and solves the equation $\sum a_r(\xi + a_r\eta)^r = 0$. By means of a float, the inclination $\sin^{-1}\eta$ is related to the depth by the relation $\eta = \xi/c$. This makes it possible to get the roots as functions of η and avoids the difficulties created by negative coefficients.

A hydrostatic machine for solving a system of simultaneous linear equations was developed by FUCHS in three papers.[20,21,22] The last and longest of these contains the important elements in the description of his machine. A system of stationary containers, partly filled with water, are arranged in rows and columns. Each row corresponds to an equation $a_i x + b_i y + \cdots = l_i$. Each column corresponds to a variable x, y, etc. The containers in any column are interconnected so that the water will rise in all of them an amount equal to the corresponding unknown: x, y, etc. In each row, hollow floating cylinders with cross-sections a_i, b_i, are fastened to a beam which is loaded with a weight proportional to l_i. The cylinders are thus depressed into their containers so that the total displaced water, nearly $a_i x + b_i y + \cdots$, is balanced by the load l_i. Provision is made for negative coefficients by letting water flow into the corresponding float and weight it down, instead of being displaced by the float.

Other variations of the same idea provide for inner and outer water levels for each term, one to be filled for positive and the other for negative

terms. Changes in level indicate the values of the variables. Another variation interchanges the role of rows and columns so that a loading l_i of water is run into the inner cylinders of one column and the values of the unknowns are read off as the weights thus applied to the bars in the several rows. Still another machine uses a system of containers in which water communicates by rows, and oil, floating on the water, communicates in columns.

More recently (1940) these hydraulic principles were applied by SCHUMANN[33] in designing a mechanism for calculating regression equations and multiple correlation coefficients, which is also adaptable to the solution of simultaneous linear equations. He estimates that a skilled computer working with slide rule and adding machine requires $5m^2 + m^3/4$ minutes to solve a problem in m variables, whereas his machine will solve the problem in $6m$ minutes. To each independent variable corresponds a horizontal beam B_j (somewhat like the plates in Wilbur's machine[17]) from which the given values y_{ij} of the unknowns y_j protrude as directed spikes in a horizontal plane. On the end of each spike is a pulley P_{ij}. A rotation of β_j in the beam for a variable y_j lifts the i^{th} pulley a distance $y_{ij} \sin \beta_j$ and releases twice that length from a cord supporting the pulley. Each equation is represented by a cord. The i^{th} cord, with one end displaced upward by an amount x_i, then comes down to engage the pulley on the spike y_{i2}, then up over another pulley and back down for the next variable y_{i3}, etc., and finally ends in a heavy float F_i suspended in mercury. If we set $k_j = 2 \sin \beta_j$, then the float F_i will be lifted an amount $x_i - \sum y_{ij}k_j$ when the beams B_j rotate through β_j and the hooks for the dependent variables are raised by x_i. The tension in each cord is proportional to that displacement. If there are more equations than independent variables, the coefficients k_j which the machine indicates are shown to be the regression coefficients. By articulating one beam at a time with the others clamped, a set of rotations γ_j can be found from which the multiple correlation coefficients can be computed with relative ease.

6. Electric and electromagnetic devices for finding real and complex roots of algebraic equations. A fascinating series of papers published in 1888 by LUCAS gives the reader an opportunity to watch an important new scientific idea being born. A first paper[34] presents a generalization of Rolle's theorem. In this paper a set of unit masses, placed at the zeros $Z_1, Z_2, \cdots, Z_p$ of a polynomial $F(z)$ in the complex plane, are assumed to repel a unit mass N at z with forces inversely proportional to the distances $|Z_j - z|$ from N. The components P and Q of the total force are given by the equation

$$P - iQ = F'(z)/F(z). \tag{6.1}$$

Roots of the derived function $f(z) = F'(z)$ are equilibrium positions for N. The orthogonal trajectories of the isodynamic lines are curves of degree $2p - 1$ (called "lignes halysiques") passing alternately through the p roots of $F(z)$ and the $n = p - 1$ roots of $F'(z)$, and are used in Lucas' generalization of Rolle's theorem.

A second paper[35] considers placing electrodes instead of masses at points $Z_1, Z_2, \cdots, Z_{n+1}$. The equipotential curves are Cassinian ovals whose nodal points occur at the n roots of the derivative $F'(z)$, which we denote by $f(z)$. In the experimental method due to GUÉBHARD,[41] a large circular conductor

is used in place of a Cassinian oval at the outer boundary of a salt solution in which a thin plate of polished metal is immersed. The points of a bundle of wires are then placed as electrodes representing as nearly as possible the points $Z_1, Z_2, \cdots, Z_{n+1}$. Colored rings deposited by electrolytic action furnish a diagram of equipotential lines whose nodal points are the n roots $z_1, z_2, \cdots, z_n$ of the derivative $F'(z)$. Such figures, obtained experimentally for the second degree binomial and the fourth degree trinomial, are displayed in the above quoted work by Guébhard.[41]

A third paper of Lucas[36] gives an electric resolution of algebraic equations with numerically given real or complex coefficients, by reducing the given equation to an equation of lower degree. An equation $f(z) = 0$ of degree n, can be solved electrically, as described above, if we know the $n + 1$ roots of the function $F(z)$, which is an integral of $f(z)$. In applying the first reduction method of Lucas we suppose some integral of $f(z)$ to be separated into even and odd functions $\phi(z^2) + z\psi(z^2)$, and we then calculate one of the $[n/2]$ roots of the polynomial ψ considered as a function of z^2. If this root be λ^2 then a particular integral of $f(z)$, namely $F(z) = \phi(z^2) - \phi(\lambda^2) + z\psi(z^2)$, is divisible by $z^2 - \lambda^2$, and the quotient is a polynomial of lower degree, $n - 1$, whose roots together with $+\lambda$ and $-\lambda$ are the required electrode points for solving $f(z)$. For example, in solving the biquadratic equation $f(z) = 5z^4 + 10z^3 + 3z^2 - 2z - 6 = 0$, we have $\psi(z^2) = z^4 + z^2 - 6$, which has the factor $z^2 - 2$. Then choosing $F(z) = (z^2 - 2)(z^2 + 3z + 2.5z^2 + 4)$ we next find three roots of the cubic factor. An integral of the cubic is $[(5/6)z^2 + 4] \times (.3z^2 + z + .36)$, of which the first factor was the "ψ" function for this cubic. Using the four roots of this factored biquadratic as electrode points, we solve the cubic electrically; then we use these three nodal points and the two points $\pm\sqrt{2}$ as five electrode points to solve the given equation electrically. An alternative method of reduction is also given, which has the disadvantage of introducing extraneous roots, and which we shall not describe here.

A fourth paper [37] discussing the electric determination of the isodynamic lines of any polynomial—which are the equimodular lines of its logarithmic derivative—describes the field of force due to a system of repulsive centers at the roots of $F(z)$, combined with a system of attractive centers at the roots of the derivative $F'(z)$.

The climax of these investigations is reached in a fifth paper,[38] in which the use of partial fractions provides LUCAS with a simple and direct procedure for locating the complex roots of a polynomial with real coefficients. An auxiliary polynomial $p(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{n+1})$ is chosen with distinct roots at convenient points L on the x-axis. Instead of working with $f(z)$ and using the roots of its integral as electrode points, Lucas now works with the function $f(z)/p(z) = \sum \mu_j/(z - \lambda_j)$ and its exponential integral

$$F(z) = (z - \lambda_1)^{\mu_1}(z - \lambda_2)^{\mu_2} \cdots (z - \lambda_{n+1})^{\mu_{n+1}}. \tag{6.2}$$

The constants (residues) $\mu_j = f(\lambda_j)/P'(\lambda_j)$ are easily computed and are introduced into the machine as charges on the fixed electrodes at the points $z = \lambda_j$. The roots of the logarithmic derivative of $F(z)$ are the required roots of the given polynomial $f(z)$, and appear as nodal points of the isodynamic

lines. Experimentally these can be deposited electrolytically on a polished metal plate, or they can be traced out by the Kirchhoff and Carey Foster method. In the perfected model of the Lucas electric equation-solver, given in a sixth paper,[39] the need for an outside conductor is eliminated by choosing $n + 2$ instead of $n + 1$ sources and sinks on the axis. The algebraic sum of the charges on the $n + 2$ electrodes is zero, and the apparatus is somewhat simpler to construct. Still another paper published two years later[40] suggests a method of introducing the coefficients into the machine by means of resistances proportional to $1/\mu_j$.

An electric device proposed by KANN[42] in 1902 was the result of his inventing a balance similar to that of Lalanne,[19] without having been at first aware of Lalanne's result. When Mehmke called this to his attention, he conceived the idea of carrying out the same principle electrically, replacing the moments of weight by variable resistances. In his mechanical balance are slits or wires in the shape of curves $\pm x$, $\pm x^2/10$, $\pm x^5/100$, etc., fastened in a horizontal plane from which are suspended weights proportional to the coefficients. These weights are constrained to lie under a movable guide slot perpendicular to the x-axis on which the plane is balanced. Values of x for which the moments are in equilibrium are the required roots. In Kann's first electrical model, each of the functions $\pm x$, $\pm x^2/10$, etc., is represented by a curve of heavy wire fastened in a removable template, which is to be mounted vertically by sliding it into one of several pairs of slots at the top and bottom of a frame. For positive terms the templates are slid in right side up, and for negative terms up side down. A sliding vertical frame in a plane perpendicular to all these templates, and perpendicular to the horizontal axis in each, serves as a root finder. The coefficients a_m are introduced by placing in the root finder, where its plane intersects the m^{th} template, a fine wire whose resistance per unit length is proportional to $|a_m|$ (multiplied if necessary by the power of 10 used in the reduction of the template graph). The current then flows through a length of this wire proportional to the ordinate, and thus through a resistance proportional to the given term. These fine wires of considerable resistance make contact with the heavy wire curves of the templates on the one hand, and with leads in the root finder frame on the other hand, and the successive templates are connected in series in such a way that the connecting resistances are either negligible or may be assumed constant and be balanced out by the bridge resistance which controls the constant term in the equation. As the root finder frame is moved in the direction of the x-axes of the templates, the roots are found at points where a bridge galvanometer which balances the equation reads zero. At these points an adjusted resistance (equal to the fixed resistances in the circuit, plus the constant resistances introduced by inverting the templates for negative variable terms) just balances a resistance for the given constant term plus all the fixed and variable resistances connected in series in the main circuit.

To avoid having to use wires of different resistances for each new set of coefficients, Kann describes in a final paragraph of the same paper a gear mechanism, one to be associated with each of the templates in the frame, whereby a resistance proportional to a given term may be unwound from a drum in a continuous manner as the root finder is moved along all the given power curves simultaneously. For each template there is in the root finder

frame a straight vertical shaft with a notch on top which presses against a rigid metal curve in the template (such as the curve $y = x^2/10$). The lower part of each of these shafts is provided with gear teeth which mesh into an exchangeable gear having a number of teeth proportional to the coefficient of the corresponding term in the equation. The motion of the shaft, thus multiplied by the coefficient $|a_m|$, is made to turn a drum mounted on a horizontal axis, around which wire is wound whose specific resistance takes care of the reduction factor of a power of 10 in the template graph. A weight hanging on the wire which unwinds from the drum maintains the upward pressure on the shaft, and as the root finder moves, a resistance is unwound which is jointly proportional to the coefficient (number of teeth) and to the power of x (ordinate of shaft top). These resistances are connected in series as in the other model, and the roots are found at the zero readings of the galvanometer. The method is limited to the determination of real roots, however.

RUSSELL & WRIGHT,[43] in an electric device constructed in 1909, combine the principle of slide rule multiplication with addition and subtraction. Multiplication is obtained from a thin insulating template in the shape of the curve $\log (y/k) = -x/n$, about which a hundred terms of no. 36 insulated manganin wire is wound, so that the wires are nearly parallel. The area in the interval $x_1 \leq x \leq x_2$ is proportional to $y_1 - y_2$. By adding a fixed resistance in series, the total variable resistance is made proportional to y_1, and the number y_1 is placed on a logarithmic scale under x_1 on the axis of abscissas. Powers of a variable x can be multiplied into the coefficient resistances by setting them on a parallel logarithmic scale. Contact fingers using a tangent scale are used to adjust separately the powers of the unknown. Then terms are added electrically by combining currents in series (or in parallel, using reciprocal logarithmic scales). Finally the real root is obtained when combined currents vanish.

Shortly after the paper of Russell and Wright, RUSSELL & ALTY[44] (1909) brought magnetism across the electric root-finding trail blazed by Lucas; inventing another machine for determining complex roots of equations, and stating that the error in its readings would not exceed 1%. In their electromagnetic method "the horizontal field due to the earth's magnetism is used in an analogous manner to the conducting sheet in Mr. Lucas' method. A drawing board with a slit cut in it, a few pieces of bell-wire, any form of 'charm' compass, ordinary ammeters and rheostats or lamp resistance boards, such as are found in every physical laboratory, can be utilized at once for the experiment."

Using our previous notation, and assuming real coefficients, let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{6.3}$$

and let $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$, where the n real numbers λ_j are subject to the condition $\sum \lambda_j = -a_{n-1}/a_n$. Then let the partial fraction expansion be $f(z)/p(z) = a_n + \sum \mu_j/(z - \lambda_j)$.

In the horizontal complex plane let the earth's magnetic flux H be directed parallel to the imaginary y-axis, so that the real x-axis is perpendicular to the magnetic meridian. Let vertical wires through the points $(\lambda_j, 0)$ carry currents C_j, producing a magnetic force $C_j/5r_j$ at the point $z = x + iy$, where $r_j = |z - \lambda_j|$. The components of resultant magnetic force at the

point z are

$$(6.4)\quad -X = \frac{C_j}{5r_j}\frac{y}{r_j}, \quad Y = H + \frac{C_j}{5r_j}\frac{x-\lambda_j}{r_j}; \quad Y + Xi = H + \sum \frac{C_j/5}{z-\lambda_j}$$

By adjusting the currents C_j to equal $5H\cdot\mu_j/a_n$, respectively, the neutral points of zero force will be precisely those for which $f(z) = 0$. Only $n-1$ ammeters and $n-1$ rheostats are required for the apparatus.

An electric calculating machine for solving simultaneous linear equations was described by MALLOCK[45] in 1933. Improving upon Mallock's experimental model of 1931, this machine for solving ten equations in ten unknowns, to within an error less than 0.1% of the largest root, was constructed by the Cambridge Instrument Co. Ltd. The machine will also give a direct solution, by least squares, of a set of equations of condition, without forming the normal equations. One closed circuit represents each equation, which is first transformed algebraically so that all coefficients are less than unity. The coefficients are then represented by the relative number of turns of the variable coils in a given circuit, each wound about the transformer corresponding to one of the unknowns. Negative signs are obtained by reversing the current in a given coil, and the answers may be read from voltmeters, attached to unit coils about the several transformers. Compensators are introduced to balance out the effect of resistances in the circuits. Repeated approximations may be used to give greater accuracy.

A recent development in the mechanical solution of algebraic equations is the use of rotating and logarithmically expanding parts to represent arguments and absolute values of complex roots of equations. In an electric machine described by HART & TRAVIS,[46] a set of $n+1$ coaxially mounted generators G_0 to G_n have their rotors rigidly connected together, but the stators of n of them are constrained by gears to rotate through $\theta, 2\theta, \cdots, n\theta$ with respect to the one on G_0 which is fixed. The alternating voltage $E\cos(\omega t + k\theta)$ on the k^{th} generator is then multipled by a_k in a coefficient potentiometer, and by Z^k in a modulus potentiometer. The latter is activated by steel tapes wrapped around a spiral cam whose arc is proportional to the k^{th} power of the angle of rotation of the cam shaft. (Only roots of modulus <1 are read directly, the others being obtained from the reciprocal equation.) The voltages $Ea_kZ^k\cos(\omega t + k\theta)$ are then added in series and connected to an indicator, the voltage on which may be written:

$$(6.5)\quad (E/2)\sum_k[(a_kZ^ke^{ik\theta})e^{i\omega t} + (a_kZ^ke^{-ik\theta})e^{-i\omega t}].$$

This vanishes, independently of t, if and only if

$$(6.6)\quad \sum a_kZ^ke^{ik\theta} = 0, \qquad \text{or} \qquad f(Ze^{i\theta}) = 0.$$

To find the roots $Ze^{i\theta}$ the angle θ is turned fairly rapidly and the modulus cam shaft slowly so that $Ze^{i\theta}$ traces a spiral in the unit circle of the complex plane. Points of zero reading are the roots, and they can be read within 2% in modulus, and 1% in argument.

The use of a cam to generate a power of a variable appears also in a later article by GREEN,[47] whose square root extractor is used to change over graphically recorded pressure differences into a graph of rates of flow. One indicator contacts the edge of a cam whose contour is $r = k\theta^2$, while the

other is driven by a pulley around the same shaft, producing a square root mechanically.

7. Methods of harmonic analysis. The isograph,[48, 49, 50] designed by T. C. FRY of the Bell Telephone Laboratories following a suggestion of A. J. KEMPNER in 1928, and the S. L. BROWN[51, 52] harmonic synthesizer-analyzer, are two modern machines for solving algebraic equations. Both solve the equation $f(re^{i\theta}) = 0$ by summing separately the sine and cosine terms in the function $f(re^{i\theta}) - c_0$, and mapping this function for fixed r as a curve in a complex plane. The isograph can handle only real coefficients, whereas the Brown analyzer as modified by Brown and Wheeler[52] can also solve equations with complex coefficients. For further details on these machines the reader is referred to the reviews which have appeared in *MTAC*.[51, 52]

8. Calculating machines. It should not be forgotten that algebraic equations of higher degree in one unknown and simultaneous linear equations in several unknowns can be solved numerically by standard methods of algebra (such as Horner's or Newton's method), using a computing machine to save labor in the processes of addition, subtraction, multiplication and division. An excellent review of the history of Calculating Machines was written by BAXANDALL.[53] One such machine, the Hamann-Automat of the Deutsche Telephonwerke und Kabelsindustrie, Aktiengesellschaft, Berlin, described by WERKMEISTER,[54] has the advantage, in solving simultaneous linear equation such as $\sum a_i x_i = 0$, $\sum b_i x_i = 0$, $\cdots$, that a quotient b_1/a_1 obtained by a division can be automatically transferred from the result register to the multiplicand register without being copied, and can then be multiplied again by each of the successive coefficients a_2, a_3, $\cdots$ in the elimination of x_1 between the first two equations.

A survey article by LILLEY,[55] previously reviewed in *MTAC*, p. 61–62, mentions not only calculating machines, but such machines as the BUSH differential analyzer.[56] Interesting as this machine is, its primary purpose is the solution of differential equations rather than algebraic equations, so we shall not describe it here.

In conclusion the author wishes to express his thanks to R. C. A. for his suggestions and help in compiling the literature, and to B. H. Bissinger for his assistance in looking up several of the references. In the footnotes which follow, those papers which have not been examined are marked in the usual manner.

J. S. FRAME

Michigan State College
East Lansing, Michigan

[1] J. A. DE SEGNER, "Methodus simplex et universalis, omnes omnium aequationum radices detegendi," Akad. Nauk, S.S.S.R., Leningrad, *Novi Commentarii Acad. Sc. Imp. Petrop.*, v. 7, 1758, p. 211–226.

[2] J. ROWNING, "Directions for making a machine for finding the roots of equations universally, with the manner of using it," R. So. London, *Trans.*, v. 60, 1770, p. 240–256.

[3] R. MEHMKE, "Numerisches Rechnen," *Encykl. d. Math. Wiss.*, v. 1, part 2, 1902, p. 1067–1073. French ed. by M. D'OCAGNE, *Encycl. d. Sci. Math.*, tome 1, v. 4, fasc. 3, 1909, p. 339–340, 432–446.

[4] R. E. MORITZ, "Some physical solutions of the general equation of the n^{th} degree," *Annals Math.*, s. 2, v. 6, 1905, p. 112–126.

[6] I. GHERSI, *Matematica dilettevole e curiosa*, (*Manuali Hoepli*), Milano 1913, p. 244–249, "Metodi fisici per la soluzione dei sistemi di equazioni algebriche."

[7] P. RIEBESELL, "Über Gleichungswagen," *Z. Math. Phys.*, v. 63, 1914, p. 256–274.

[8] J. L. LAGRANGE, *Leçons élémentaires sur les Mathématiques*, 1795. English translation: *Lectures on Elementary Mathematics*, Chicago, Open Court, 1898, p. 124–126.

[9] E. LILL, "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," Acad. d. Sci., Paris, *Comptes Rendus*, v. 65, 1867, p. 854–857; *Nouv. Annales Math.*, s. 2, v. 6, 1867, p. 359–362.

[10] L. CREMONA, *Calcolo Grafico*, Turin, 1874. English translation: *Graphical Statics*, Oxford, 1890, p. 70–76.

[11] H. CUNYNGHAME, "On a mechanical method of solving quadratic and cubic equations, whether the roots be real or impossible," *Phil. Mag.*, s. 5, v. 21, 1886, p. 260–263.

[12] R. MEHMKE, "Über einen Apparat zur Auflösung numerischer Gleichungen mit vier oder fünf Gliedern," *Z. Math. Phys.*, v. 43, 1898, p. 338–340.

[13] A. B. KEMPE, "On the solution of equations by mechanical means," *Messenger Math.*, s. 2, v. 2, 1873, p. 51–52.

[14] T. H. BLAKESLEY, "Logarithmic lazytongs and lattice-works," *Phil. Mag.*, s. 6, v. 14, 1907, p. 377–381.

[15] T. H. BLAKESLEY, "A kinematic method of finding the roots of a rational integral equation . . . ," *Phil. Mag.*, s. 6, v. 23, 1912, p. 892–900.

[16] R. F. MUIRHEAD, "A mechanism for solving equations of the n^{th} degree," Edinburgh Math. So., *Proc.*, v. 30, 1912, p. 69–74.

[17] M. NÄBAUER, "Vorrichtung zur Auflösung eines linearen Gleichungssystems," *Z. Math. Phys.*, v. 58, 1910, p. 241–246.

[18] J. B. WILBUR, "The mechanical solution of simultaneous equations," Franklin Inst., *J.*, v. 222, 1936, p. 715–724.

[19] o J. B. BÉRARD, *Opuscules Mathématiques*, Paris, 1810.

[20] L. L. C. LALANNE, "Description d'une nouvelle machine à calcul pour résoudre les équations numériques des sept premiers degrés," Acad. d. Sci., Paris, *Comptes Rendus*, v. 11, 1841, p. 859–860; and report on this machine by A. Cauchy *et al.*, p. 959–961.

[21] É. C. R. COLLIGNON, *Traité de Mécanique*, v. 2, *Statique*, Paris, 1873, p. 347–349, 401; second ed. 1886; fourth ed. 1903.

[22] o K. EXNER, *Über eine Maschine zur Auflösung höherer Gleichungen*, Progr. Staatsgymnasium, IX. Bezirk, Vienna, 1881.

[23] C. V. BOYS, "On a machine for solving equations," *Phil. Mag.*, s. 5, v. 21, 1886, p. 241–245+plate III.

[24] G. B. GRANT, "Machine for solving numerical equations," *Amer. Machinist*, v. 19, 1896, p. 824–826.

[25] R. SKUTSCH, "Über Gleichungswagen," *Z. Math. Phys.*, v. 47, 1902, p. 85–104.

[26] W. PEDDIE, "A mechanism for the solution of an equation of the n^{th} degree," Int. Cong. Math., *Proc.*, Cambridge, 1912, p. 399–402.

[27] A. DEMANET, "Résolution hydrostatique de l'équation du troisième degré," *Mathesis*, v. 18, 1898, p. 81–83.

[28] G. MESLIN, "Sur une machine à résoudre les équations," *J. de Phys.*, s. 3, v. 9, 1900, p. 339–343. Also Acad. d. Sci., Paris, *Comptes Rendus*, v. 130, 1900, p. 888–891.

[29] A. EMCH, "Two hydraulic methods to extract the n^{th} root of any number," *Amer. Math. Mo.*, v. 8, 1901, p. 10–12, 58–59.

[30] J. MASSAU, "Mémoire sur l'intégration graphique et ses applications," Assoc. d. Ingénieurs sortis d. Écoles Spéciales de Gand, *Annales*, v. 2, 1878, p. 13–15, 203–281 (figs. 1–65); v. 7, 1884, p. 53–132 (figs. 66–104); v. 10, 1886, p. 1–535+12 plates (figs. 105–310). Reprinted. "Appendice," v. 12, 1889, p. 185–457+6 plates (figs. 311–350); reprinted, Paris, Gauthier Villars, 1890, 264 p. This is a complete list of references to Massau's work; the particular pages discussing the apparatus in question were not identified. The reference given by *Encycl. d. Sci. Math.*[3], p. 436, to "v. 10 (1886/7), p. 58 et planche hors texte fig. 26" is certainly incorrect.

[31] K. FUCHS, "Näherungsweise Elimination durch Mittelwerte," *Archiv. Math. Phys.*, s. 3, v. 17, 1910, p. 103–105; *Jahrb. Fort. Math.*, v. 41, p. 197.

[32] o K. FUCHS, "Eine Gleichungsmaschine aus kommunizierenden Gefassen," *Oesterr. Z. f. Vermessungsw.*, v. 10, 1912, p. 325–329; *J. Fort. Math.*, v. 43, p. 151.

[33] K. FUCHS, "Hydrostatische Gleichungsmaschinen," *Z. Math. Phys.*, v. 63, 1914, p. 203–214.

[34] T. E. W. SCHUMANN, "The principles of a mechanical method for calculating regression equations and multiple correlation coefficients and for the solution of simultaneous linear equations," *Phil. Mag.*, s. 7, v. 29, 1940, p. 258–273.

[35] F. LUCAS, "Généralisation du théorème de Rolle," Acad. d. Sci., Paris, *Comptes Rendus*, v. 106, 1888, p. 121–122.

[36] F. LUCAS, "Détermination électrique des racines réelles et imaginaires de la dérivée d'un polynôme quelconque," *loc. cit.*, p. 195–197.

[36] F. LUCAS, "Résolution électrique des équations algébriques," *loc. cit.*, p. 268–270.
[37] F. LUCAS, "Détermination électrique des lignes isodynamiques d'un polynôme quelconque," *loc. cit.*, p. 587–589.
[38] F. LUCAS, "Résolution immédiate des équations au moyen de l'électricité," *loc. cit.*, p. 645–648.
[39] F. LUCAS, "Résolution des équations par l'électricité," *loc, cit.*, p. 1072–1074.
[40] F. LUCAS, "Résolution électromagnétique des équations," Acad. d. Sci., Paris, *Comptes Rendus*, v. 111, 1890, p. 965–967.
[41] o J. C. JAMIN, *Cours de Physique de l'École Polytechnique*, Paris, v. 4, fourth ed. by E. BOUTY, 1888–91, p. 170, figs. 2–3. Also P. É. A. GUÉBHARD, *J. d. Physique*, s. 2, v. 1, 1882, p. 205.
[42] L. KANN, "Zur mechanischen Auflösung von Gleichungen. Eine elektrische Gleichungs-Maschine," *Z. Math. Phys.*, v. 48, 1902, p. 266–272.
[43] A. RUSSELL & A. WRIGHT, "The Arthur Wright electrical device for evaluating formulae and solving equations," *Phil. Mag.*, s. 6, v. 18, 1909, p. 291–308.
[44] A. RUSSELL & J. N. ALTY, "An electromagnetic method of studying the theory of and solving algebraical equations of any degree," *Phil. Mag.*, s. 6, v. 18, 1909, p. 802–811.
[45] R. R. M. MALLOCK, "An electrical calculating machine," R. So. London, *Proc.*, v. 140A, 1933, p. 457–483.
[46] H. C. HART & I. TRAVIS, "Mechanical solution of algebraic equations," Franklin Inst., *J.*, v. 225, 1938, p. 63–72.
[47] H. D. GREEN, "Square root extractor," *Rev. Sci. Instruments*, n.s., v. 11, 1940, p. 262–264.
[48] R. L. DIETZOLD, "The isograph—a mechanical root-finder," *Bell Laboratories Record*, v. 16, 1937, p. 130–134.
[49] R. O. MERCNER, "The mechanism of the isograph," *loc. cit.*, p. 135–140.
[50] "Mechanical aids to mathematics: Isograph for the solution of complex polynomials," *Electronics*, v. 11, Feb., 1938, p. 54.
[51] S. L. BROWN, "A mechanical harmonic synthesizer-analyzer," Franklin Inst., *J.*, v. 228, 1939, p. 675–694; reviewed in *MTAC*, p. 127.
[52] S. L. BROWN & L. L. WHEELER, "A mechanical method for graphical solution of polynomials," Franklin Inst., *J.*, v. 231, 1941, p. 223–243; reviewed in *MTAC*, p. 128.
[53] D. BAXANDALL, "Calculating machines," *Encyclopædia Britannica*, fourteenth ed., v. 4, 1929, p. 551–553. Note bibliography at end.
[54] P. WERKMEISTER, "Die Auflösung eines Systems linearer Gleichungen mit Hilfe der Rechenmaschine 'Hamann-Automat'," *Z. f. Instrumententechnik*, v. 51, 1931, p. 490.
[55] S. LILLEY, "Mathematical machines," *Nature*, v. 149, 1942, p. 462–465; reviewed in *MATC*, p. 61.
[56] V. BUSH, "The differential analyzer. A new machine for solving differential equations," Franklin Inst., *J.*, v. 212, 1931, p. 447–488.

References to unreviewed items are as follows:

[57] G. ROSÉN, "Eine elektromechanische 'Gleichungswage,'" *Elektrotechn. Z.*, v. 50, 1929, p. 1726–1727.
[58] G. REVESSI, "Verso soluzioni meccaniche ed elettriche dei sistemi di equazioni lineari," *L'Elettrotecnica*, v. 12, 1925, p. 550–553.

Zeros of Certain Bessel Functions of Fractional Order

The following tables contain the zeros of $J_\nu(x)$ for $x \leqslant 25$, where $\nu = \pm 1/3, \pm 2/3, \pm 1/4, \pm 3/4$. These zeros were obtained by inverse interpolation in a thirteen-place manuscript of these functions, computed by the NYMTP. The accuracy of the zeros to 10D is guaranteed, and the two additional places have a high probability of being correct.

s	$j_{+1/4,s}$	$j_{-1/4,s}$
1	2.78088 77239 95	2.00629 96717 90
2	5.90614 26988 43	5.12306 27427 46
3	9.04238 36635 83	8.25795 11756 42
4	12.18134 15289 55	11.39646 76969 87
5	15.32136 98260 12	14.53629 98843 38
6	18.46192 72456 89	17.67675 35868 47
7	21.60278 44489 13	20.81754 94222 32
8	24.74382 77961 28 (24.740)	23.95855 34952 86 (23.955)

s	$j_{+1/4,s}$	$j_{-1/4,s}$
1	3.49100 83741 08	1.05850 82594 04
2	6.65263 55231 22	4.28405 38127 24
3	9.80161 23591 40	7.44045 44040 05
4	12.94703 48891 39	10.58817 91486 60
5	16.09096 95281 99	13.73311 84505 74
6	19.23414 17604 82	16.87681 75138 75
7	22.37687 15748 16 (22.384)	20.01985 75839 86
8		23.16250 59340 75 (23.169)

s	$j_{+1/3,s}$	$j_{-1/3,s}$
1	2.90258 62484 17	1.86635 08588 74
2	6.03274 70572 66	4.98785 32314 35
3	9.17050 66694 64	8.12426 53819 40
4	12.31019 37716 45	11.26351 48254 28
5	15.45064 89678 17	14.40377 58801 36
6	18.59148 63361 81	17.54451 06557 21
7	21.73254 11617 47	20.68550 48061 24
8	24.87373 14228 06 (24.871)	23.82665 62470 57 (23.824)

s	$j_{+1/2,s}$	$j_{-1/2,s}$
1	3.37561 06526 94	1.24304 62596 19
2	6.53025 59365 13	4.42912 06776 99
3	9.67658 06352 38	7.57945 84465 30
4	12.82060 86784 66	10.72474 69244 99
5	15.96368 38809 06	13.86837 45833 31
6	19.10627 35045 92	17.01125 45001 33
7	22.24858 23933 60 (22.253)	20.15373 45371 51
8		23.29597 58670 60 (23.300)

For $x > 25$, ten or more decimal places in the zeros may be obtained from the well-known formula (five terms) for the roots of Bessel functions, given below; see, for example, G. N. WATSON, *Treatise on the Theory of Bessel Functions*, 1922 and 1944, p. 506; the sixth term was supplied by W. G. BICKLEY,[1] *Phil. Mag.*, s. 7, v. 34, 1943, p. 40:

$$j_{\nu,s}=\beta-\frac{\mu-1}{2^3\beta}-\frac{(\mu-1)(7\mu-31)}{3\cdot 2^7\cdot\beta^3}-\frac{(\mu-1)(83\mu^2-982\mu+3779)}{15\cdot 2^{10}\cdot\beta^5}$$
$$-\frac{(\mu-1)(6949\mu^3-153855\mu^2+1585743\mu-6277237)}{105\cdot 2^{15}\cdot\beta^7}$$
$$-\frac{(\mu-1)(70197\mu^4-24\,79316\mu^3+480\,10494\mu^2-5120\,62548\mu+20921\,63573)}{315\cdot 2^{18}\cdot\beta^9}-\cdots,$$

where $\beta = (s + \frac{1}{2}\nu - \frac{1}{4})\pi$; $\mu = 4\nu^2$. The first five terms of the above expression will yield at least 10 decimals for roots greater than those given here. The values of β corresponding to the highest roots given here are noted in parenthesis; it is apparent that for x close to 25, β approximates the root to two decimals at least.

The author desires to express his appreciation of assistance rendered by several members of the NYMTP in checking these values.

MILTON ABRAMOWITZ

NYMTP

[1] It may be noted that in Bickley's article the fifth term of the formula has an erroneous number 6277327, for 6277237.

RECENT MATHEMATICAL TABLES

169[B].—BAASMTC, *Mathematical Tables, Volume IX. Table of Powers giving integral powers of integers. Initiated by* J. W. L. GLAISHER, *extended by* W. G. BICKLEY, C. E. GWYTHER, J. C. P. MILLER, E. J. TERNOUTH. Edited by Miller. Cambridge University Press, 1940, xii, 132 p. 21.5 × 27.9 cm. 15 shillings.

NYMTP, *Table of the First Ten Powers of the Integers from 1 to 1000.* New York, December 1, 1938 (title page), 1939 (outside cover), 80 p. 21 × 35 cm. Reproduced in mimeographed form. Originally listed at $.50; out of print. Not to be reprinted.

These two tables, published within a year of one another, are similar in their purpose, to provide computers with an adequate table of the higher powers of the integers. The first, however, covers a much greater range than the second, although for powers in which the two overlap the second is somewhat easier to use because of the various printing devices which were used to condense the longer work.

The BAASMTC tables originated with a work prepared by J. W. L. GLAISHER more than 70 years ago to which the following reference was made in the 1873 Report of the Committee on Mathematical Tables: "Mr. J. W. L. Glaisher has had formed in duplicate a table giving the first twelve powers of the first thousand numbers, which, after the calculation has been made independently a third time, will be stereotyped and published, probably in the course of 1873; it is hoped that it will help to make the tabulation of mathematical functions less laborious and difficult." But the tables never appeared.

Intrigued by this reference and also by a reference to these tables made in 1905 by A. J. C. CUNNINGHAM, L. J. COMRIE initiated a diligent search for a copy. His efforts were finally rewarded by success and in 1935 H. J. WOODALL presented to the Committee the copy to which Cunningham had made reference. This is a proof copy and is probably the only one in existence. The introduction to the work under review states that "without this copy it is doubtful if the production of the present volume would have been undertaken."

With the original Glaisher computation as a nucleus, the scope of the table was extended to give x^n for the following ranges:

$n = 2(1)12$	$x = 1(1)1099$
$n = 13(1)20$	$x = 1(1)299$
$n = 21(1)50$	$x = 1(1)120.$

Because of the limitations of space, the powers corresponding to $n = 28$ and 29 were curtailed, and for values of n greater than 30, except for $n = 40$ and $n = 50$, the powers were also curtailed. In these abbreviated values the last figure printed is always the true value and is not raised. For convenience in use the number of figures retained is a multiple of five, and the counting of the digits is assisted by small figures printed at the top and bottom of the columns or in the margins. In the incomplete values never fewer than 21 figures are retained; below 100 not more than 25 are given, but in the range to 120 the number may be as great as 28.

"The tables are arranged so that, whenever possible, an opening contains powers of a hundred consecutive integers, and so that consecutive powers of a given integer are normally on the same or adjacent openings. . . . The complete range of powers for each hundred consecutive integers forms a single 'chapter' which is completed before the next is started."

For compactness in printing various devices have been adopted. Thus, we find that "powers have in some cases been combined in pairs so as to give an approximately constant total number of digits. For example, in the first 'chapter,' x^9 and x^{20} appear together on pages 4 and 5, followed by x^{10} and x^{19} on pages 6 and 7, and so on, up to pages 24 and 25, which contain x^{19} and x^{30}. Thus for consecutive powers of, say, 57, we turn forward until we

come to x^{30}, then backward (as indicated by a footnote) until x^{20} is reached; another footnote directs us to page 27 for x^{21}, and from there we proceed forward to x^{50}."

Unusual care was taken to insure accuracy in the table. For this purpose the following formula was used,

$$\sum_{x=1}^{X} x^n = \frac{1}{n+1} X^{n+1} + \tfrac{1}{2} X^n + a_1 X^{n-1} - a_2 X^{n-3} + a_3 X^{n-5} - \cdots,$$

where we abbreviate:

$$a_{2r+1} = \frac{n(n-1)(n-2)\cdots(n-r)}{(2r+2)!} B_{r+1},$$

in which B_{r+1} is the $(r+1)$th Bernoulli number. The powers were summed in blocks of 50 and these totals compared with those computed directly from the formula. The figures were also read against those given in other existing tables where the ranges overlapped. This resulted in the detection of a number of errors in other tables, which have been listed in the "Bibliography."[1] No errors were discovered in the calculations of Glaisher, nor in the NYMTP tables, which were loaned by A. N. Lowan for this check.

In addition to the table of powers there is given a table of the binomial coefficients[2] $\binom{n}{r}$ for $n = 2(1)50, r = 2(1)10$ and a table of the numerators and denominators of the coefficients a_{2r+1} for values of $n \leqq 50$.

The NYMTP table was the first publication of that Project. It was computed from the first four powers of the integers given in P. Barlow's *Table of Squares, Cubes. . . .* Because of unusual precautions taken to avoid errors in the final stencils, it was believed that the table was without error. The check by the BAASMTC, which we have mentioned above, confirmed this belief.

H. T. D.

[1] Errors in the following works are listed:

P. Barlow, *New Mathematical Tables . . .*, London, 1814

K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen . . .*, Berlin, 1926

K. Hayashi, *Fünfstellige Funktionentafeln . . .*, Berlin, 1930

K. Hayashi, *Tafeln für die Differenzenrechnung . . .*, Berlin, 1933

K. Pearson, *Tables for Statisticians and Biometricians*, Part II, London, 1931

H. W. Weigel, $x^n + y^n = z^n$? *Die elementare Lösung des Fermat-Problems . . .*, Leipzig, 1933

L. Zimmermann, *Vollständige Tafeln der Quadrate aller Zahlen bis* 100 009, third ed., Berlin, 1938.

[2] A complete table of the binomial coefficients $\binom{n}{r}$ for $n = 2(1)50$, $r = 2(1)50$, is contained in J. W. L. Glaisher, *Messenger Math.*, s. 2, v. 47, 1917, p. 97–107.

170[B].—*Square Root Divisors*, 1941, 2 p., *Square Root Multipliers*, 1940, 2 p., *Cube Root Divisors*, 1944, 2 p. *Publications* MM 56, 57, 68 of the Marchant Calculating Machine Co., Oakland, California. 21.5 × 28 cm.

These tables are one-page substitutes for more extensive tables of square and cube roots and are supposed to be used in connection with any standard computing machine.

The table of Square Root Divisors, T. 56, gives 8-place values of $2A^{\frac{1}{2}}$ and $2(10A)^{\frac{1}{2}}$ for 166 integral values of A (at irregular intervals) between 100 and 1000. The table of Square Root Multipliers, T. 57, gives for the same values of A the reciprocals of the values in T. 56.

To use T. 56 to find an approximation to the square root of a given number N, we enter the table at the value of A which is nearest to N and take out the value of $2A^{\frac{1}{2}}$ (or perhaps $2(10A)^{\frac{1}{2}}$). Then the approximation

$$N^{\frac{1}{2}} = \frac{N + A}{2A^{\frac{1}{2}}}$$

is obtained by machine division of $N + A$ by the tabulated entry. Thus the method is based on the fact that the arithmetic and geometric means of two nearly equal numbers are very nearly equal.

T. 57 merely replaces the operation of division by multiplication. Probably the average computer will find T. 56 a bit more efficient than T. 57 when his machine has automatic division and the quotient can be copied down while the machine is operating. In some cases when the square root is in the denominator T. 57 might prove more effective.

It may be argued that no table for square root is necessary if one has a computing machine. Most computers are familiar with a square root process based on the identity

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

which is well suited to some machines and which will yield the square root of an occasional number sooner than the operator can locate a table. Tables 56 and 57 will be found more efficient when a great many scattered square roots to 5S are needed quickly.

The Table of Cube Root Divisors, T. 68, gives 6-place values of $3A^{2/3}$, $3(10A)^{2/3}$ and $3(100A)^{2/3}$ for 162 values of A between 100 and 1000. It is used in a similar way to obtain cube roots by the approximation

$$N^{1/3} = \frac{N + 2A}{3A^{2/3}}.$$

The irregular intervals of the argument A of the three tables are so chosen that one application of the process will give the required root correct to at least 5S, no matter what value of N is given. On the reverse sides of T. 56 and 68 are examples of the re-application of the appropriate formulae a second time to obtain approximations correct to 9S and 10S.

The method involved in these tables was apparently suggested by comments of L.J.C. in his "Introduction" to the third edition of *Barlow's Tables* (1930). The general method, which is usually superior to interpolation, may be used with any table of square or cube roots, or indeed any table of squares or cubes. T. 68, however, is decidedly more convenient for cube roots.

The reviewer has two photostatic copies of a 3-page manuscript table of the same sort for 5th roots prepared and supplied by the same company. No doubt this useful table will be published in due course, to replace *Basic Publication* MM 88, of Aug., 1940, 2 p.

Quite another method for cube and 5th roots has been given by Dederick.[1] A one-page table enables the computer to obtain either root to 10S.

D. H. L.

[1] L. S. DEDERICK, "A modified method for cube roots and fifth roots," *Amer. Math. Mo.*, v. 33, 1926, p. 469–472. See also D. H. LEHMER, "On the use of the calculating machine for cube and fifth roots," *Amer Math. Mo.*, v. 32, 1925, p. 377–379. In the "table of cube and fifth roots" for this latter article are the following slips: $\sqrt[5]{10}$, *for* 93193, *read* 93192; $\sqrt[5]{1000}$ *for* 71705, *read* 71706; $\sqrt[5]{1.02}$ *for* 27709, *read* 22710; $\sqrt[5]{5}$, *for* 29662, *read* 29661.

171[D].—A. LINNEBACH, "Mehrkreisige Siebenschaltungen mit ausgeglichener Resonanzkurve," *Elek. Nach. Technik*, v. 20, Oct. 1943, p. 247–248. 21.7 × 28 cm.

There are tables of exact values, and to 6D, of $\cos \frac{(2\nu - 1)}{2n}\pi$, and $\sin \frac{(2\nu - 1)}{2n}\pi$, $n = 1, 2$, $\nu = 1$; $n = 3, 4$, $\nu = 1, 2$; $n = 5, 6$, $\nu = 1(1)3$; $n = 8$, $\nu = 1(1)4$; $n = 10$, $\nu = 1(1)5$; $n = 12$, $\nu = 1(1)6$. Also exact values, and to 5D, of $\cos \frac{2\nu - 1}{24}\pi$, $\sin \frac{2\nu - 1}{24}\pi$, $\nu = 1(1)6$.

172[E].—NATIONAL DEFENSE RESEARCH COMMITTEE, Division 6, *Tables of the Bipolar Transformation*, compiled by M.I.T. Underwater Sound Laboratory, Report, November 1944, 13 leaves + 3 plates (2 folding). 21.5 × 28 cm. Printed by the photo-offset process from manuscript.

These tables are available only to certain Government agencies and activities.

In all transmission line problems the transformation

$$\tanh [\pi(\alpha + i\beta)] = R + iX = \rho e^{i\phi}$$

is of importance.

The tables give values of R(5D), X(5D), ρ(4D), and ϕ (to the nearest 1′) as functions of $\alpha = 0(.01).2(.02).4(.05).6(.1)1$, and $\beta = 0(.02).5$.

The following formulae are useful:

$$R = \frac{\sinh (2\pi\alpha)}{\cosh (2\pi\alpha) + \cos (2\pi\beta)}, \qquad X = \frac{\sin (2\pi\beta)}{\cosh (2\pi\alpha) + \cos (2\pi\beta)},$$

$$R^2 + [X + \cot (2\pi\beta)]^2 = \csc^2 (2\pi\beta),$$

$$[R - \coth (2\pi\alpha)]^2 + X^2 = \operatorname{csch}^2 (2\pi\alpha).$$

The first plate (27.8 × 34.3 cm.) is a graph of $\tanh [\pi(\alpha + i\beta)] = \rho e^{i\phi}$, $\phi = 0(0^\circ.5)90^\circ$, $\rho = .1(.01)1(.1)10$, $\alpha = .01(.01).06(.02).2(.05).3(.1).5$, $\beta = .01(.01).06(.02).2(.05).3(.02)$-$.44(.01).5$.

The second plate (27 × 27.8 cm.) shows contours of the transformation $R + iX = \tanh [\pi(\alpha + i\beta)]$, $R = 0(.05)5$, $X = 0(.05)5$, $\alpha = .005(.005).1(.01).25$, $.3(.02).4(.1).6$, $\beta = 0(.02).1$, $.15(.01).4(.05).5$. There are three errors in marking the letters α and β, and 7 places where I is substituted for X.

The third plate (27.8 × 19.1 cm.) displays contours for the equation $\rho e^{i\phi} = \sinh [\pi(\alpha + i\beta)]$, $\rho = .1(.005).2(.01)1(.05)2(.1)8$, $\phi = 0(2^\circ)90^\circ$, $\alpha = .02$, $.05(.05).3(.1).9$, $\beta = .02$, $.05(.05).5$.

173[E].—A. PUCHER, "Rechteckplatten mit zwei eingespannten Rändern," *Ingenieur-Archiv*, v. 14, 1943, p. 250–251. 19.3 × 26.8 cm.

There are 8 tables, to 5S, for $x = .25(.05)3.6$, of the following functions:

$$\frac{\sinh \pi x + \pi x \cosh \pi x}{\frac{1}{2}\sinh 2\pi x + \pi x}, \quad \frac{\cosh \pi x + \pi x \sinh \pi x}{\frac{1}{2}\sinh 2\pi x - \pi x}, \quad \frac{\pi x \sinh \pi x}{\frac{1}{2}\sinh 2\pi x + \pi x},$$

$$\frac{\pi x \cosh \pi x}{\frac{1}{2}\sinh 2\pi x - \pi x}, \quad \frac{\sinh \pi x}{\frac{1}{2}\sinh 2\pi x \pm \pi x}, \quad \frac{\cosh \pi x}{\frac{1}{2}\sinh 2\pi x \pm \pi x}.$$

174[E].—P. B. WRIGHT, "Resistive attenuator, pad and network, theory and design" part 2 of a 4-part paper, *Communications*, v. 24, Oct. 1944, p. 64, 66, 68. 19.6 × 27.1 cm.

Tables for 20 log $e^\theta(e^{2\theta} = k^2 \geq 1) = 0(.01).2(.05).4(.1)4(.5)30(1)60(5)140$, 150 of (a) $\sinh \theta = (k^2 - 1)/2k$; (b) $\cosh \theta = (k^2 + 1)/2k$; (c) $\tanh \theta$; (d) $\tanh \frac{1}{2}\theta = (k - 1)/(k + 1)$; (e) $\cosh^2 \theta$; (f) $\sinh^2 \frac{1}{2}\theta = (k - 1)^2/4k$; (g) $\operatorname{csch} \theta$; (h) $\operatorname{sech} \theta$; (i) $\coth \theta$; (j) $\coth \frac{1}{2}\theta$; (k) $\operatorname{sech}^2 \theta$; (l) $\operatorname{csch}^2 \frac{1}{2}\theta$. These tables are to 5–9S.

175[I].—A. N. LOWAN & H. E. SALZER. "Coefficients for interpolation within a square grid in the complex plane." *J. Math. Phys.*, M.I.T., v. 23, 1944, p. 156–166. 17.5 × 25.5 cm.

The usual theory of polynomial interpolation holds in the complex domain as well as on the real axis. Thus, given an analytic function $f(z)$ of the complex variable $z = x + iy$ and $n + 1$ arbitrary points $z_0, z_1, \cdots, z_n$ within the domain of regularity of $f(z)$, there will be one, and only one, polynomial $\phi_n(z)$ of nth degree which will assume the prescribed values $f(z_k)$ for $k = 0, \cdots, n$. The degree of approximation of $\phi_n(z)$ to $f(z)$ can be judged from Taylor's formula. The polynomial $\phi_n(z)$ can be represented in many forms, but we are

here concerned only with the standard Lagrange form

$$\phi_n(z) = \sum_{k=0}^{n} \frac{P_\nu(z)}{P_\nu(z_k)} f(z_k),$$

where $P_\nu(z) = (z - z_0) \cdots (z - z_{\nu-1})(z - z_{\nu+1}) \cdots (z - z_n)$.

Originally, this formula has been regarded as a purely theoretical tool. However, in the real domain tables of the coefficients $P_\nu(z)/P_\nu(z_k)$ are now available for the typical case of equally spaced z_k. With such tables[1] one has only to perform the $n + 1$ multiplications by $f(z_k)$, and the products are accumulated in any modern computing machine without clearing. Thus, using tables and a modern computing machine, the Lagrange interpolation formula will lead to the required value $\phi_n(z)$ without any writing and auxiliary computations. In the complex plane the situation is different only in that a multiplication of two complex numbers requires four ordinary multiplications.

The present paper provides tables of the quadratic and cubic Lagrange interpolation polynomials if the three or four given points are vertices of a typical square: z_0, $z_1 = z_0 + h$, $z_{-1} = z_0 + ih$, $z_2 = z_0 + h + ih$ (h real), and it is desired to interpolate for a point $z = z_0 + \Delta z$ within the square. In the usual way one puts $\Delta z = h(p + iq)$ so that $0 \leqq p \leqq 1$, $0 \leqq q \leqq 1$. We have then to interpolate at the point $P = p + iq$ of the unit square. The quadratic and cubic formulae are written in the form

$$\phi_2(z) = L_{-1}^{(2)}(P)f(z_{-1}) + L_0^{(2)}(P)f(z_0) + L_1^{(2)}(P)f(z_1)$$

and

$$\phi_3(z) = L_{-1}^{(3)}(P)f(z_{-1}) + L_0^{(3)}(P)f(z_0) + L_1^{(3)}(P)f(z_1) + L_2^{(3)}(P)f(z_2).$$

The seven coefficients $L_{-1}^{(3)}(P)$ etc. are tabulated (p. 160–166), each in the form of a double-entry table with the arguments p and q varying from 0 to 1 in steps of .1. The values are exact, which means that the coefficients $L_k^{(2)}$: $L_{-1}^{(2)}(P)$, $L_1^{(2)}(P)$ are given to 3D and $L_0^{(2)}(P)$ to 2D, while the coefficients $L_k^{(3)}(P)$, $k = -1(1)2$, are given to 4D.

W. Feller

Brown University

[1] NYMTP, *Tables of Lagrangian Interpolation Coefficients*, New York, 1944.

176[I].—H. E. Salzer, "Table of coefficients for inverse interpolation with central differences," *J. Math. Phys.*, M.I.T., v. 22, 1943, p. 210–224. 17.5 × 25.5 cm.

The author has done a genuine service to practical interpolation by tabulating not only the coefficients for inverse interpolation by central differences, as given in the article under review, but also for a similar tabulation of the coefficients of the formula with advancing differences; see the review by D. H. L., *MTAC*, p. 315 f. The difficult problem of finding inverse values from tables which are computed to arguments that are not sufficiently close to assure linear interpolation, is thus made much simpler by means of these tables.

The tables computed by Salzer are based upon the inverse of what is commonly called the Laplace-Everett formula for interpolation. If $f(x)$ is the function tabulated for $x = x_0$, $x_0 + h$, $x_0 + 2h$, etc., and if p lies between 0 and 1, then the Laplace-Everett formula may be written:

$$(1)\quad f(x + ph) = f(x) + p\Delta f(x) - [E_2(q)\delta^2 f(x) + E_2(p)\delta^2 f(x + h)] + [E_4(q)\delta^4 f(x) + E_4(p)\delta^4 f(x + h)] - \cdots$$

where $q = 1 - p$, $\delta^2 f(x)$, $\delta^4 f(x)$ etc. are central differences, and

$$E_2(p) = p(1^2 - p^2)/3!, \qquad E_4(p) = p(1^2 - p^2)(2^2 - p^2)/5! \text{ etc.}$$

If we make the further abbreviations:

$$m = \frac{f(x + ph) - f(x)}{\Delta f(x)}, \qquad d_0^r = \frac{\delta^r f(x)}{\Delta f(x)}, \qquad d_1^r = \frac{\delta^r f(x + h)}{\Delta f(x)},$$

then equation (1) can be written in the form:

$$p = m + E_3(q)d_0^3 + E_3(p)d_1^3 - E_4(q)d_0^4 - E_4(p)d_1^4 + \cdots.$$

This formula can now be inverted by means of Lagrange's formula for the inversion of functions of the form $p = m + \lambda\phi(p)$, subject to the condition $|\lambda\phi(p)| < r$, where $r = |p - m|$. This inversion leads to the following expansion:

$$p = m + {}_mC_3 d_0^3 + \tfrac{1}{6}m(1 - m^2)d_1^3 + A(m)d_0^4 + B(m)d_1^4 + C(m)(d_0^2)^3 + \cdots,$$

where ${}_mC_3$ is the third binomial coefficient and the functions $A(m)$, $B(m)$, $C(m)$, etc. are polynomials in m. If one defines a product of the form $(d^p)^r(d^q)^s$ as being a term of order $pr + qs$, then the inversion is carried out through terms of sixth order. The original calculation of the coefficients by H. T. D.[1] were checked and one printing error detected, the first number in the coefficient of the term $d_0^3(d_1^2)^2$ being 1 instead of 2.

Exclusive of the coefficients of the terms of second order there are 15 coefficients to be tabulated, which the author has designated by $A(m)$, $B(m)$, $\cdots$, $O(m)$. The first two coefficients ${}_mC_3$ and $m(1 - m^2)/6$ are to be found in the recently published *Tables of Lagrangian Interpolation Coefficients* of NYMTP (*MTAC*, p. 314f), and are accordingly omitted by the author. The remaining 15 coefficients are computed to 10D. Those of fourth order, namely $A(m)$ through $E(m)$ are given over the range $m = 0(.001)1$ and the author makes use of the relationships:

$$A(m) = B(1 - m), \qquad -C(m) = D(1 - m), \qquad E(m) = -E(1 - m),$$

to reduce the space occupied by the tables. The coefficients of sixth order are given over the range 0(.1)1.

H. T. D.

[1] H. T. Davis, *Tables of the Higher Mathematical Functions*, v. 1, Bloomington, Ind., 1933, p. 82–83.

177[J].—H. Stenzel, "Das Schallfeld eines Strahlers in einer Mediumschicht mit schallweicher und schallharter Begrenzung," *Ann. d. Phys.*, s. 5, v. 43, June, 1943, p. 30, 18. 13.8 × 21.5 cm.

$$\text{Table of } \psi(x) = \frac{1}{2x} - \frac{1}{1^2 - x^2} + \frac{2}{2^2 - x^2} - \frac{3}{3^2 - x^2} + \cdots$$

for $x = [.005(.005).5; 4D]$.

On p. 18 $iH_0^{(1)}(x)$ is given, to $4D$, for $x = .6084, .4156, .4712i, .9024i, 1.266i, 1.610i, 1.943i, 2.272i, 2.595i, 2.918i, 3.238i, 3.558i, 3.876i, 4.194i, 4.512i, 4.829i, 5.145i, 5.462i, 5.794i, 6.094i$.

178[L].—P. K. Bose, "On confluent hypergeometric series," *Sankhyā. The Indian J. Statistics*, v. 6, Feb., 1944, p. 407–412. 22.5 × 29.2 cm.

For integral and half integral values of a and c the confluent hypergeometric series ${}_1F_1(a; c; x)$ occurs in F-Statistic under non-null hypothesis, in Studentized D^2-Statistic and in the expression for the multiple correlation coefficient for a particular type of parent population. Tables with 5S are given here for $a = 2(1)15$ and with 4S for $a = 16(1)25$. The values chosen for c are 1, 2, 3, 4 and the values of x are 2, 3, 3.8, 5.7, 7, 8.4, 9, 10.5, 13.5. A short table is given also to show the agreement between the value calculated from the series and the value calculated by recurrence formulae starting from the known values for $a = 2, 3$; $c = 1, 2$.

These new tables include some values of x not considered in previous tables. In those of Webb and Airey[1] for $c = 1(1)7$, $a = -3(.5)4$ the range with 4S was $x = 1(2)6(1)10$. In Airey's first table[2] for $2c = -3(2)3$, $2a = -8(1)8$ the range with 6S was $x = 0(.1)1(.2)3(.5)8$. In the tables of Gran Olsson[3] c has the values 1, 2, 3 but a is not an integer, 4D being given. In the tables of Chappell[4] $c = 1$, $a = \frac{1}{2} - k$, $k = 1(1)10$, $kx = .1(.1)1.5, 2(1)10$ with 4D.

The functions for which c is an integer occur in many physical problems some of which are mentioned by Webb and Airey.[1] The Bessel function $I_n(z)$ in which n is an integer or half an odd integer is related to these functions, so also are the polynomials of Laguerre and Sonin. The case $c = 3$ occurs in the work of SEN[5] on stresses in some rotating circular disks of varying thickness while the function ${}_1F_1(a; n + 2; -2ky)$ occurs in the work of LAMB[6] on atmospheric oscillations, and some calculations were impracticable on account of the lack of tables. The polynomials of Hermite correspond to the case in which $2c$ is 1 or 3. Some tables of these functions are already available. The author indicates that further numerical results are forthcoming.

Some numerical results relating to the roots of Laguerre polynomials $L_n(x) = F(-n; 1; x)$ suitable for numerical integration over the range $(0, \infty)$ have been given by KOSHLIAKOV.[7] Tables of the function

$$H(m, a, x) = \exp(-ix)F(m + 1 - ia; 2m + 2; 2ix),$$

with 6D, have been published by LOWAN & HORENSTEIN[8] for $m = 0(1)3$, $a = 0(1)10$, and $x = 0(1)10$. The 2 in $2ix$ has been omitted on the cover of the reprints.

H. B.

[1] H. A. WEBB and J. R. AIREY, "The practical importance of the confluent hypergeometric function," *Phil. Mag.*, s. 6, v. 36, 1918, p. 129–141.

[2] J. R. AIREY, "The confluent hypergeometric function," B.A.A.S., *Report*, 1926, p. 276–294; 1927, p. 220–244.

[3] R. GRAN OLSSON, (a) "Biegung kreisförmiger Platten von radial veränderlicher Dicke," (b) "Tabellen der konfluenten hypergeometrischen Funktion erster und zweiter Art," *Ingenieur-Archiv*, v. 8, 1937, p. 81–98, 99–103.

[4] G. E. CHAPPELL, "The properties of a new orthogonal function associated with the confluent hypergeometric function," Edinburgh Math. So., *Proc.*, v. 43, 1925, p. 117–130.

[5] B. SEN, "Note on the stresses in some rotating circular disks of varying thickness," *Phil. Mag.*, s. 7, v. 19, 1935, p. 1121–1125.

[6] H. LAMB, "On atmospheric oscillations," R. So. London, *Proc.*, v. 84A, 1910, p. 551–572.

[7] N. S. KOSHLIAKOV, "O vychislenii po formule mekhanicheskikh kvadratur opredelennykh integralov s beskonechnymi predelami" [On the calculation of integrals to infinite limits by means of formulae of mechanical quadratures], Akad. Nauk, Leningrad, *Izvestiia*, s. 7, *Fiziko-matematicheskoe otdelenie*, v. 7, 1933, p. 801–808.

[8] A. N. LOWAN & W. HORENSTEIN, "On the function $H(m, a, x) = \exp(-ix)F(m+1-ia; 2m+2; 2ix)$," *J. Math. Phys.*, M.I.T., v. 21, 1942, p. 273–283.

179[L].—BAASMTC, *Mathematical Tables, Volume VI. Bessel Functions, Part I, Functions of Orders Zero and Unity.* Cambridge, University Press, 1937, xx, 288 p. 21.5 × 28 cm. Compare *MTAC*, p. 282f. £2.

The circular cylinder is the most symmetrical body after the sphere. This is perhaps the reason why Bessel functions which arise in the solution of boundary problems for domains with circular symmetry are the most widely used transcendental functions after the circular functions which enter in the solution of boundary problems for domains with spherical symmetry. From the standpoint of the frequency of occurrence, the Bessel functions of orders zero and unity are undoubtedly the most important. Moreover, the recurrence formula between Bessel functions of three consecutive integral orders make it a relatively simple matter to generate the values of Bessel functions of integral orders from those of orders zero and unity. These considerations serve to stress the great contribution to science of the tabular volume before us, devoted to Bessel functions of orders zero and unity. This volume may truly be said to have disposed of the tabulation of the Bessel functions of orders 0 and 1 for all time.

A description of the volume will serve to prove the truth of this statement. The Bessel functions $J_0(x)$ and $J_1(x)$ are solutions of the differential equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ for $n = 0$ and $n = 1$ respectively, which are finite at $x = 0$. The bulk of the volume under review (170 out of 288 pages) is devoted to the tabulation of these functions. Ten-place values of $J_0(x)$ and $J_1(x)$ and second central differences are tabulated at intervals of .001 in the range from 0 to 16 and at intervals of .01 in the range from 16 to 25. Beyond $x = 25$,

the auxiliary functions $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ appearing in the asymptotic expansions $J_0(x) = A_0(x) \sin x + B_0(x) \cos x$; $J_1(x) = B_1(x) \sin x - A_1(x) \cos x$ are tabulated in the range from 25 to 1150 at intervals (ranging from .1 to 10) so chosen that second central differences are adequate for the maximum attainable accuracy. In the range from 1000 to 6000 the auxiliary functions are tabulated at intervals of 100 together with "modified" second central differences. Beyond $x = 6000$, one term each in the expansions of the auxiliary functions suffices to generate values to eight-place accuracy. Thus to all intents and purposes the volume under review disposes of the tabulation of $J_0(x)$ and $J_1(x)$ over the entire range of the argument from 0 to infinity. The situation is quite similar with the other Bessel functions tabulated in the volume with some exceptions to be discussed.

The functions of the second kind $Y_0(x)$ and $Y_1(x)$ have a logarithmic singularity at the origin and a table of these functions would not be interpolable near the origin unless the interval is very small. For this reason the authors have tabulated 8-place values of the functions $Y_0(x) - \frac{2}{\pi} J_0(x) \ln x = C_0(x)$ and $x\left\{ Y_1(x) - \frac{2}{\pi} J_1(x) \ln x \right\} = C_1(x)$ in the range from 0 to .5 at intervals of .01. The functions $D_0(x) = \frac{2}{\pi M} J_0(x)$ and $D_1(x) = \frac{2}{\pi M} J_1(x)$, where $M = \log e$, are also tabulated alongside the values of $C_0(x)$ and $C_1(x)$. With the aid of the four auxiliary tables just described, one may compute $Y_0(x) = C_0(x) + D_0(x) \log x$ and $Y_1(x) = C_1(x)/x + D_1(x) \log x$. As is the practice throughout the entire volume, second central differences are tabulated with the values of the auxiliary functions in question.

In addition to these functions the values of $Y_0(x)$ and $Y_1(x)$ are also given in the range from 0 to .5 with or without modified second central differences. Beyond $x = .5$ and through $x = 25$, eight-place values are given (with ordinary or modified second central differences) for arguments at intervals of .01. Beyond $x = 25$, the auxiliary functions $A_0(x)$, $B_0(x)$, $A_1(x)$ and $B_1(x)$ above mentioned may be used for the computation of $Y_0(x)$ and $Y_1(x)$ by means of the relations $Y_0(x) = B_0(x) \sin x - A_0(x) \cos x$ and $Y_1(x) = -A_1(x) \sin x - B_1(x) \cos x$.

The Bessel functions $I_0(x)$ and $I_1(x)$ are solutions of the differential equation $x^2y'' + xy' - (x^2 + n^2)y = 0$, for $n = 0$ and $n = 1$ respectively, which remain finite at $x = 0$. Eight-place values of these functions and second central differences are given in the range from 0 to 5 at intervals of .001. Beyond $x = 5$, the related functions $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are tabulated at intervals of .01 from $x = 5$ to $x = 10$ and at intervals of .1 from $x = 10$ to $x = 20$.

The functions $K_0(x)$ and $K_1(x)$, the second fundamental solutions associated with $I_0(x)$ and $I_1(x)$, have a logarithmic singularity at the origin and therefore for the sake of interpolability the functions $E_0(x) = K_0(x) + I_0(x) \ln x$ and $E_1(x) = x\{K_1(x) - I_1(x) \ln x\}$ are tabulated together with the products $F_0(x) = -\frac{1}{M} I_0(x)$ and $F_1(x) = \frac{1}{M} I_1(x)$ for x ranging from 0 to .5 at intervals of .01. In addition $K_0(x)$ and $K_1(x)$ are given for x ranging from 0 to 5 at intervals of .01. Beyond $x = 5$, the values of $e^xK_0(x)$ and $e^xK_1(x)$ are given alongside the values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ in the same range and for the same interval as the last-named function.

To round out the description of the volume under review, let it be mentioned that it contains the first 150 zeros of $J_0(x)$ and $J_1(x)$ and the corresponding values of each of the functions at the roots of the other; the first 50 zeros of $Y_0(x)$ and $Y_1(x)$ together with the corresponding values of $Y_1(x)$ and $Y_0(x)$ at these zeros, a one-page table of e^x and e^{-x}, a one-page table of the Everett coefficients of the second difference, and a 2½ page table of the Besselian coefficients of the double second difference.

In addition to a general introduction devoted to a description of the tables and of their preparation, each of the first eight major tables is preceded by a page giving the definition of the functions, recurrence relations, and other pertinent information. The typography of the volume is excellent and as far as accuracy is concerned the reviewer has the feeling

after reading the Introduction that the authors were fully justified in their remark that "There is every reason to believe that the tables are completely free from error."

ARNOLD N. LOWAN

180[L].—FEDERAL TELEPHONE AND RADIO CORP., *Reference Data for Radio Engineers*, New York, 1943 (fourth printing Aug. 1944), p. 197–200. 13.8 × 21.6 cm.

Here are 6 tables of Bessel functions: T. 1–2, $J_0(x)$ and $J_1(x)$, x = [0(.1)15.9; 4D]; T. 3–5, $J_n(x)$, n = 2, 3, 4, x = [0(.1)4.9; 4D]; T. 6, $J_n(p)$, p = 1(1)14, n = [0(.5)10; 4S].

181[L].—NYMTP, "Table of $f_n(x) = \dfrac{n!}{(x/2)^n} J_n(x)$", *J. Math. Phys.* M.I.T., v. 23, 1944, p. 45–60. 17.5 × 25.3 cm.

Under BAASMTC 2 (*MTAC*, p. 283) reference was made to elaborate ms. tables of $J_n(x)$ with the following ranges:

$$n = 0(1)20, \quad x = [0(.1)25; 10D]; \quad n = 0(1)12, \quad x = [0(.01)10; 8D].$$

Though these tables were made available to the authors of this table of $f_n(x)$, they have preferred to recompute and publish a new table, since although $J_n(x)$ may be given to 8 or 10 decimals, over a considerable range there are but one or two significant figures, whereas the function here adopted has all nine decimals significant over almost the whole range, particularly for n large, where the defect of the J_n table is most noticeable.

The first tables of these functions were calculated by A. WALTHER, S. GRADSTEIN & K. HESSENBERG and published in JAHNKE & EMDE, second ed., 1933, p. 250–258, with the notation

$$\Lambda_n(x) \text{ [J \& E]} = f_n(x)\text{[NYMTP]}, \quad \text{for} \quad n = 0(1)8, \quad x = [0(.02)9.98; 4\text{–}5D].$$

There appears no advantage in changing from the symbol Λ introduced by Emde, to the symbol f, which may even cause confusion as it is commonly needed for a general functional symbol. The symbol Λ will therefore be used in this review.

This table is not only of great value for the actual numerical values thereby made available, but also is of theoretical interest. The functions selected for tabulation and the method of starting with the largest values of n and working down to small n, are clearly the means by which nature intended that Bessel Functions should be computed.

The $\Lambda_n(x)$ functions have also a valuable property perhaps not envisaged by the authors, opening up possibilities of double-entry tables interpolable in n as well as in x. If curves be plotted of $\Lambda_n(x)$ against n for a fixed parameter x[1(1)10], it is seen that for $n > x$ or thereabouts the curves (monotonic, asymptotic to 1 for $n = \infty$) suggest this possibility. To test this, values of $\Lambda_n(5)$, n = 5(1)20, extracted from the tables and differenced, were interpolated for n = 12.5, 13.5, 14.5 using 8th difference Everett (the 8th differences only affect a unit in the 9th (last) decimal). Compare my inefficient efforts to interpolate $J_n(1)$ between n = 1 and 2 to 7D, requiring 6 differences, although fundamental values had been laboriously computed at interval .1 [*MTAC*, p. 99, where the interval erroneously given as .01 was later corrected, p. 132]. The three values at n = 12.5, 13.5, 14.5 satisfied the recurrence relation $\Lambda_{n-1}(x) = \Lambda_n(x) - [x^2\Lambda_{n+1}(x)]/4n(n+1)$; and repeated applications thereof yielded $\Lambda_{2.5}(5)$ = .08083 8726.

Now $\dfrac{25}{3}\Lambda_{2.5}(5) = \left(\dfrac{5\pi}{2}\right)^{\frac{1}{2}} J_{2.5}(5) = \left(\dfrac{3}{5^2} - 1\right) \sin 5 - \dfrac{3}{5} \cos 5$, on making the obvious substitution of $\Gamma(n+1)$ for $n!$ when n is not integral. Taking sin 5 and cos 5 from BAASMTC, *Mathem. Tables*, v. 1, and computing the true value of $\Lambda_{2.5}$, the result is .08083 87261, with which the value obtained by interpolation agrees so far as it goes. Hence it appears that the table can be interpolated for fractional n to nearly the full accuracy of the original table, provided we start from $n > 2x$ or thereabouts.

So it seems that it would be possible, without excessive labour, to derive a double-entry table for $n = 0(.1)20$, $x = [0(.1)10; 7D]$, that would be interpolable in both n and x without going further than second differences at most. Since $\Lambda_n(x)$ is an even function of x, it would be unnecessary to tabulate for x negative (compare $J_n(x)$, for fractional real n and negative real x, complex). But difficulties arise for Λ_n when n is negative, since $\Lambda_n = \infty$ for n a negative integer. Even conversion to J_n is of little practical use here [$J_n(.1)$ fluctuates from zero to -38 and back, between $n = -1$ and -2]. The only practical solution would appear to be interpolation (if required) in the region n positive, followed by the use of recurrence formulae.

In equation (3), p. 46, the n in the denominator of the last fraction should be deleted so as to read $x^2/4(n+1)(n+2)$.

C. R. COSENS

Engineering Laboratory
University of Cambridge

182[L].—G. N. WATSON, *A Treatise on the Theory of Bessel Functions.* Second edition, Cambridge: at the University Press, New York: The Macmillan Co., 1944, viii, 804 p. 18.7 × 23.2 cm. $15.00.

This offset print of the first edition of 1922 seems to have been in the press for more than three years since Watson's preface to the second edition is dated "March 31, 1941." It is as follows (apart from acknowledgments of assistance):

"To incorporate in this work the discoveries of the last twenty years would necessitate the rewriting of at least chapters XII–XIX; my interest in Bessel functions, however, has waned since 1922, and I am consequently not prepared to undertake such a task to the detriment of my other activities. In the preparation of this new edition I have therefore limited myself to the correction of minor errors and misprints and to the emendation of a few assertions (such as those about the unproven character of Bourget's hypothesis) which, though they may have been true in 1922, would have been definitely false had they been made in 1941."

Hence the numbers of pages in the first and second editions are the same. In the Bibliography, p. 753–788, the only change seems to be an addition of a title on p. 788. We note that equations (3), (4), p. 81 have been corrected. In *MTAC*, p. 307, we have already listed the tables in this work, p. 666–752. The errors which we there noted in T. I–II have now vanished but many others still remain; see *MTAC*, p. 296, and *MTE* **58, 60**, where a beginning has been made in listing such errors. This reprint has filled a great need.

R. C. A.

183[M].—H. W. LINDEMANN, "Innenbackenbremsen," *Automobiltechnische Z.*, v. 46, Aug. 1943, p. 367–372; also English transl., "Internal shoe brakes," *Engineers' Digest* (Amer. ed.), v. 1, Sept. 1944, p. 558–563. 19.7 × 27.2 cm. English ed., v. 9, Aug., 1944, p. 230–235.

On p. 562 is a three-place table of eight integrals, from 0 to θ, of (a) $\sin t \cos t$; (b) $\sin^2 t$; (c) $\cos^2 t$; (d) $\cos t$; (e) $\sin t$; (f) $\sin^2 t \cos t$; (g) $\sin t \cos^2 t$; (h) $\sin^3 t$, for $\theta = 0(2^g)180^g$.

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **169** (BARLOW, HAYASHI, WEIGEL, L. ZIMMERMANN), **170** (LEHMER), **172** (N.D.R. COMM.), **176** (DAVIS), **181** (NYMTP); N26 (AGETON).

55. J. M. BATES, "Zeros of a class of polynomials associated with Bateman's k-function," Iowa State College *J. Sci.*, v. 12, 1938, p. 474.

The first 5 zeros of $J_{1/3}[(2/3)x^{3/2}] + J_{-1/3}[(2/3)x^{3/2}]$ are here given as follows:

2.338107, 4.137258, 5.520555, 6.786701, 7.944136.

These values should be:

2.338107, 4.087949, 5.520560, 6.786708, 7.944134.

Bates states that he calculated his zeros from those for the function $J_{1/3}(x) + J_{-1/3}(x)$, given by WATSON, *Bessel Functions*, 1922 [and 1944], p. 751; but, as noted below, Watson's second zero is highly erroneous.

J. C. P. MILLER

56. A. R. FORSYTH, *Messenger Math.*, s. 2, v. 50, 1921, p. 129–149. Compare *MTAC*, p. 215, $IB_6 6$.

This paper gives exact values of certain sums extending over the roots of Bessel functions.

P. 145, *for* $\sum_{r=1}^{\infty} \mu_r^{-6} J_0^{-2}(\mu_r) = 73/(5 \cdot 27 \cdot 4096)$,

read $\sum_{r=1}^{\infty} \mu_r^{-3} J_0^{-2}(\mu_r) = 73/(5 \cdot 27 \cdot 4096)$;

p. 146, *for* $\sum_{r=1}^{\infty} \mu_r^{-6} J_0^{-2}(\mu_r) = -37/(3^3 \cdot 128)$, $\sum_{r=1}^{\infty} \mu_r^{-6} J_0^{-2}(\mu_r) = -379/(3 \cdot 5^3 \cdot 2048)$,

read $\sum_{r=1}^{\infty} \{\mu_r^{-4} + 8\mu_r^{-6}\} J_0^{-2}(\mu_r) = -37/(3 \cdot 128)$,

$$\sum_{r=1}^{\infty} \{\mu_r^{-4} + 8\mu_r^{-6}\} J_0^{-2}(\mu_r) = -379/(3 \cdot 5 \cdot 2048).$$

To the list on p. 139 we may add the following:

$$\sum_{r=1}^{\infty} \kappa_r^{-1} J_1^{-1}(\kappa_r) = \tfrac{1}{4},$$

though this is not obtainable by the author's method.

D. H. L.

57. K. KARAS, "Tabellen für Besselsche Funktionen erster und zweiter Art mit den Parametern $\nu = \pm 2/3, \pm 1/4, \pm 3/4$," *Z. f. angew. Math. u. Mech.*, v. 16, 1936, p. 249–251. It has been already noted (*MTAC*, p. 295) that NYMTP discovered that in half of these tables, those for $J_\nu(x)$, there are 108 errors, namely: 90 of a unit in the last decimal place, and 18 of two or more units. These errors are as follows:

$\nu = 2/3$

x	*for*	*read*	x	*for*	*read*
.1	.15011	.15012	2.1	.52905	.52906
.2	.23723	.23722	4.2	−.28517	−.28518
.8	.54534	.54535	4.8	−.36057	−.36058
1.2	.62892	.62893	5.0	−.35712	−.35713
1.6	.62955	.62956	7.9	.27822	.27823
1.7	.61777	.61778			

$\nu = -2/3$

x	*for*	*read*	x	*for*	*read*
.1	2.72981	2.72976	3.0	−.45760	−.45759
.2	1.68082	1.68084	8.5	−.21775	−.21774
.4	.96247	.96246	8.6	−.23186	−.23185
.5	.76835	.76834	8.9	−.25916	−.25915
.8	.37691	.37690	9.3	−.25895	−.25894
2.6	−.48311	−.48310	9.6	−.23233	−.23232

$\nu = 1/4$

x	for	read	x	for	read
.4	.7143	.7144	6.9	.2548	.2547
1.6	.5803	.5804	7.0	.2681	.2680
1.7	.5383	.5384	7.2	.2861	.2860
2.3	.2443	.2444	7.3	.2907	.2906
3.3	−.2189	−.2188	7.4	.2925	.2923
4.3	−.3836	−.3835	7.5	.2913	.2910
4.4	−.3789	−.3788	7.6	.2873	.2869
4.7	−.3436	−.3437	7.7	.2805	.2800
4.9	−.3046	−.3045	7.8	.2711	.2704
5.5	−.1345	−.1346	7.9	.2591	.2582
5.7	−.0684	−.0685	8.0	.2449	.2436

$\nu = -1/4$

x	for	read	x	for	read
.2	1.4310	1.4319	7.1	.2742	.2743
.4	1.1560	1.1559	7.3	.2415	.2416
2.9	−.3668	−.3669	7.4	.2219	.2220
3.3	−.4227	−.4226	7.5	.2003	.2004
3.5	−.4243	−.4242	7.6	.1770	.1771
6.4	.3018	.3019	7.7	.1522	.1523
6.5	.3069	.3070	7.8	.1263	.1264
6.9	.2967	.2968	7.9	.0994	.0995
7.0	.2868	.2869	8.0	.0719	.0720

$\nu = 3/4$

x	for	read	x	for	read
.8	.4986	.4987	5.7	−.2723	−.2722
1.2	.5988	.5989	6.1	−.1693	−.1694
1.3	.6115	.6116	6.7	.0145	.0146
1.5	.6224	.6225	6.8	.0447	.0449
1.9	.5888	.5889	6.9	.0741	.0743
3.1	.1719	.1718	7.0	.1024	.1025
3.2	.1274	.1273	7.3	.1778	.1779
3.3	.0830	.0829	7.4	.1990	.1991
3.7	−.0855	−.0856	7.5	.2180	.2182
4.0	−.1934	−.1935	7.6	.2347	.2348
4.1	−.2243	−.2244	7.7	.2488	.2489
4.2	−.2524	−.2525	7.8	.2602	.2604
4.7	−.3439	−.3440	7.9	.2689	.2692
5.2	−.3482	−.3481	8.0	.2749	.2752

$\nu = -3/4$

x	for	read	x	for	read
.7	.3277	.3234			
.8	.2194	.2193	3.9	−.1509	−.1508
1.9	−.4211	−.4212	4.1	−.0719	−.0718
2.1	−.4672	−.4673	5.3	.2941	.2942
2.5	−.5008	−.5009	5.7	.3306	.3305
3.1	−.4204	−.4205	6.3	.2889	.2890
3.2	−.3943	−.3945	6.5	.2526	.2527
3.3	−.3656	−.3657	7.1	.0998	.0999
3.5	−.3007	−.3006	7.9	−.1257	−.1258

See RMT **72**, *MTAC*, p. 8.

58. E. C. J. v. LOMMEL, Bayer. Akad. d. Wissen., *math. natw. Abt., Abh.*, v. 15, 1886, p. 648; also G. N. WATSON, *A treatise on the Theory of Bessel Functions*, 1922 and 1944, p. 744; see RMT **182** and *MTAC*, p. 296. Tables of $S(2x/\pi)^{\frac{1}{2}}$ and $C(2x/\pi)^{\frac{1}{2}}$:

	$S(2x/\pi)^{\frac{1}{2}}$			$C(2x/\pi)^{\frac{1}{2}}$	
x	*for*	*read*	x	*for*	*read*
0.2	.023788	(Watson .023720 correct)	11	.380390	.380392
1.5	.415348	.415483	11.5	.395149	.395152
6.5	.347099	.347100	15.0	.569335	.569336
8.0	.512010	.512009	15.5	.524009	.524010
11.0	.504784	.504786	17.5	.406589	.406590
11.5	.447809	.447810	20.5	.587849	.587848
12.0	.405810	.405811	21.0	.573842	.573841
13.5	.432489	.432488	21.5	.542266	.542265
15.5	.598183	.598184	25.5	.526896	.526897
20.5	.504875	.504874	34.0	.537026	.537027
21.0	.545885	.545884	34.5	.504881	.504882
24.0	.467029	.467028	38.5	.545560	.545559
28.5	.573060	.573059	47.5	.479313	.479311
34.0	.557490	.557489	48.5	.443930	.443929
36.5	.476871	.476872	The 14 errors for $x < 20$ were noted by comparison with the tables of J. R. AIREY, in B.A.A.S., *Report*, 1926, p. 274–275.		
39.5	.513690	.513689			
40.5	.558799	.558800			
44.5	.447720	.447721			

J. W. WRENCH, JR.

4211 Second St., N. W.
Washington 11, D. C.

59. NYMTP, *Tables of Circular and Hyperbolic Sines and Cosines*, 1939. See RMT **89**, p. 45f.

The integral part in the value of sinh x corresponding to the argument 1.8190, on p. 364, should be 3 instead of 2.

A. N. LOWAN

60. G. N. WATSON, *Bessel Functions* . . . , p. 751. See RMT **182**, p. 364.

s_n, zeros of $J_{-1/3}(x) + J_{1/3}(x)$
s_2, *for* 5.6101956, *read* 5.5101956.

J. C. P. MILLER

UNPUBLISHED MATHEMATICAL TABLES

Reference has been made to an unpublished table in RMT **168** (TERRILL & SWEENY).

29[C].—GEORGE HASTINGS HEPPEL (1793–1845), *Table of Logarithms*, mss. and stereotyped plates possibly in possession of the Institute of Actuaries.

The principal tables in these mss. consist of the logarithms, to 12D, of 100 000 to 118 600, stereotyped in quarto; of 131 000 to 136 636; and of 200 000 to 210 000, to 15D as far as 202 024, and to 12D thereafter. Mr. Heppel was at various times an accountant and an actuary. Having decided shortly before he died that he had sufficient material to publish he got as far as having stereotyped plates made for the range indicated above. The accuracy of the tables has not been tested. The author's son has told us that the logarithms were constructed on the basis of Hutton's logarithms of numbers to 20 places. Mainly through the influence of A. DEMORGAN the tables and plates were purchased and placed in the hands of CHARLES JELLICOE, later president of the Institute of Actuaries. It would seem as if all that is known of the tables and their author is given in an article by Jellicoe, "Heppel's logarithms," Institute of Actuaries, *J.*, v. 10, 1862, p. 82–84.

R. C. A.

30[C].—JOHN THOMSON (1782–1855), *Table of Twelve-Figure Logarithms of Numbers from* 1 *to* 120 000, mss. presented to the Royal Astronomical Society in 1873 by his sister CATHERINE THOMSON. Mss. in the R.A.S. Library, though not listed in its printed *Catalogue* of 1886 or 1925.

All that is known about these tables is contained in an "Account" drawn up at the request of the Council of the R.A.S. by J. W. L. Glaisher, and published in R.A.S., *Mo. Notices*, v. 34, 1874, p. 447–475. A few extracts from the information thus supplied are now given.

The tables in about a dozen volumes include a complete 12-place table of the logarithms of numbers from unity to 120 005 (besides the logarithms of 744 higher numbers at irregular intervals, the highest being log 123 187) to 12D; first and second differences for numbers from 29 561 to 120 001; and logarithms of numbers from unity to 4 000 to 15D. Another volume containing second differences for numbers from 10 000 to 29 560 has probably been lost. Here are wholly original tables.

To determine the accuracy of the mss. Glaisher read parts of them with existing tables giving more than 12 figures. One such table is Briggs, *Arithmetica Logarithmica*, London, 1924 with 14-place logarithms of numbers from 1 to 20 000 and from 90 000 to 101 000. It was found that most of the ms. errors were in end-figures. By comparison with a number of other tables Glaisher assembled many interesting results.

Little is known of Thomson except that he was born at Strachur, Argyleshire, and was the second in a family of eleven children of a farmer. He was educated at Strachur till he was twelve years of age, when he went to Greenock, and was placed under the care of a teacher of mathematics and navigation, from whom he imbibed a taste for mathematical studies. For many years he occupied the position of clerk to a firm in Greenock, but subsequently he commenced business as an accountant.

31[C, D, Q].—*Arithmetic, Logarithmic, Trigonometric, and Astronomical Tables*, computed, 1848, 1869–89, by EDWARD SANG, and his daughters JANE NICOL SANG, FLORA CHALMERS SANG, and presented in 1907 to the Royal Society of Edinburgh, in custody for the British Nation.

These manuscripts, now at the R.S.E., consist of 47 volumes which were originally accompanied by transfer duplicates of 33 of the volumes; these latter volumes were deposited in the Library of the University of Edinburgh. Of the 47 volumes, 26 (1–6, 12–19, 28–30, 39–47) are the work of E.S., 5 (7, 32, 33, 36, 37) of J.N.S., and 16 (8–11, 20–27, 31, 34, 35, 38) of F.C.S. The chief tables in these volumes are as follows:

The first 10 000 Prime Numbers (up to 104 729) with their logarithms to 28D. Computed 1848, 1865–75.

Logarithms of Numbers 100 000–370 000, to 15D, Δ^2. Computed 1869–1873? Also the last 10 figures of the logarithms of numbers from 1 00000 0000 to 1 00000 9999, and from 1 00000 0000 to 99999 0000. Computed 1884.

Sines for each centesimal 5 minutes of the quadrant, to 33D, Δ^3. Computed 1877. See *MTAC*, p. 36f.

Log Sines for every .0001 radian up to .25, and log [arc/sine], to 7D, with Δ. Computed 1880–85.

Sines and cosines for each centesimal minute in the quadrant, to 15D, Δ^2. Computed 1881.

Log Sines for each centesimal minute in the quadrant, to 15D, 0 to 50^s, Δ; 50^s to 100^s, Δ^2. Log Tangents, for each centesimal minute, to 15D, 0 to 50^s, Δ. Computed 1885–88; revised 1889.

Sines for each centesimal minute in the quadrant, to 8D, Δ^2. Computed 1879.

The tables so far listed, along with their computations, are in v. 1–44. Then follow:

Tables of Circular Segments and Mean Anomalies. Writing the equation for Kepler's problem of finding the mean anomaly (m) of a planet's orbit from the eccentric anomaly (u)

in the form

$$m = u - e \sin u = u - \cos\theta \sin u$$
$$= \tfrac{1}{2}\{(u + \theta) - \sin(u + \theta)\} + \tfrac{1}{2}\{(u - \theta) - \sin(u - \theta)\},$$

it appears that if u and θ are known, m may be derived from a table which gives the quantity $\frac{1}{2}(x - \sin x)$ with argument x. This quantity is the area of the segment of angle x^c in a circle of unit radius. Sang first calculated a table of circular segments for each centesimal minute in the 400^s, to 8D, Δ^3. The results are given in terms of the unit one grade of area, the sector of a circle in which the arc subtends one grade; hence the whole circle contains 400^g or units of area.

Thus were derived the corresponding mean anomalies, to 8D, by a double reference, for each arc $u = 0(1^g)\ 200^g$, and each degree of arc of ellipticity, $e = 0(1^g)\ 99^g$.

Such matters are discussed, and some printed tables to 4D are given by Sang, in his "Nouveau calcul des mouvements elliptiques," Accad. Sci., Turin, *Mem.*, v. 32, 1880, p. 187–199, 305–307. The table of segments is given p. 192–193; and the tables of anomalies for $e = \cos 10^g$, $\cos 50^g$, $\cos 90^g$, p. 194–199.

And in v. 47 "the Anomalies are given only to the nearest second, but the differences for a change of 1^g of position, and the variations for a change of 1^g in ellipticity, are filled in; and thus, of the three—the eccentricity, the position, the anomaly—any one may be determined from the others." The previous table to 8D is here cut to 4D, with Δ.

In this same volume are also a Table of all Rational Right-angled Triangles having the hypotenuse less than 1000, arranged according to the magnitude of the hypotenuse; and a Table of Other Triangles whose sides ($\leqslant 1000$), and areas, are integers.

Greater details concerning these manuscripts, and ideas developed in them, may be found in

1. "Dr. Edward Sang's logarithmic, trigonometrical, and astronomical tables," R. So. Edinburgh, *Proc.*, v. 28, 1908, p. 183–196. Reprinted, with a portrait photo of Sang, in E. M. Horsburgh, *Modern Instruments and Methods of Calculation*, London, 1914, p. 38–47.

2. C. G. Knott, "Edward Sang and his logarithmic calculations," *Napier Tercentenary Memorial Volume*, London, 1915, p. 261–268; see also R. A. Sampson, p. 236–237.

3. E. Sang, "On the need for decimal subdivisions in astronomy and navigation, and on tables requisite therefor," R. So. Edinburgh, *Proc.*, v. 12, 1884, p. 533–544.

4. E. Sang, "On the construction of the canon of logarithmic sines," *idem*, p. 601–619; see also v. 9, 1878, p. 343–352.

5. E. Sang, "Notice of fundamental tables in trigonometry and astronomy, arranged according to the decimal division of the quadrant," R. So. Edinburgh, *Proc.*, v. 16, 1889, p. 249–256.

For some facts we are also indebted to Henderson (see *MTAC*, p. 2).

There is a biography of Edward Sang (1805–1890) by D. B. Peebles in R. So. Edinburgh, *Proc.*, v. 21, 1897, p. xvii–xxxii; this includes a list of his writings. Born in Kirkcaldy, Scotland, he was a pupil of Edward Irving (Scottish church divine and friend of Thomas Carlyle) until 1818, when he went to Edinburgh and entered the University. With the exception of two years (1841–1843) spent in Manchester New College as professor of mechanical science, and of twelve years (1843–1854) spent in Turkey, where he assisted in establishing schools of civil engineering, and in laying out railroads, he lived and worked in Edinburgh as a private teacher of mathematics. For many years he was secretary of the Royal Scottish Society of Arts. In 1883 the University of Edinburgh conferred on him the degree of LL.D., and in 1884 he was elected an honorary member of the Franklin Institute in Philadelphia. His first published work was a large folio volume, *Life Assurance and Annuity Tables* with *a copious collection of rules and examples*, Edinburgh, 1841, second v. 1859. For interesting comments on this work, see A. De Morgan, *English Cyclopædia*, ed. C. Knight, *Arts and Science* section, v. 7, London, 1861, col. 1013.

He was the first one to publish a logarithmic table beyond 108 000, *A New Table of Seven-Place Logarithms of all numbers from* 20 000 *to* 200 000, . . . *prepared under the auspices of the managers of the Associated Life Insurance Offices in Scotland*, London, 1871, second

edition, improved, Edinburgh, 1878. A set of stereotype plates for this work was given to the R.S.E. with Sang's manuscripts. See J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, 1926, p. 135f; E. SANG, "Account of the new table of logarithms to 200 000," R. So. Edinburgh, *Trans.*, v. 26, 1872, p. 521–528. Also *Specimen Pages of a table of the Logarithm of all Numbers up to One Million . . . shortened to nine figures from original calculations to fifteen places of Decimals*, [1874?], 26 p. 4to, not in the List of his Writings. There is a copy in the Brown University Library. The first page is a copy of a memorial of the president and council of the R.S.E. dated 19 June 1874, to the Chancellor of Her Majesty's Exchequer, and the last 16 pages are devoted to communications from academies and scholars, dated 1872–73, commending the idea of publishing a 9-place table of logarithms of numbers to 10^8, in three volumes; but the publication was never achieved.

Sang "received a grant of £100 per annum from [the] Government as a recognition of his valuable scientific work; and the associated Scottish life assurance offices, feeling that some substantial recompense was due to him for his logarithms and actuarial tables, at a meeting in 1878 resolved to recommend to the offices the payment of an annuity of £100 . . . for the remainder of his life, which was agreed to, and subscribed by the offices."

R. C. A.

MECHANICAL AIDS TO COMPUTATION

12[Z].—L. J. COMRIE, "Mechanical computing," Appendix I, p. 462–473 of DAVID CLARK, *Plane and Geodetic Surveying*, v. 2, third ed. revised by JAMES GLENDENNING, London, Constable, 1943. 14 × 21.7 cm. Also as a pamphlet reprint.

Some computations possible on several calculating machines are described and there are illustrations of four of them, namely: (1) Brunsviga, (2) 10-key Facit, (3) Marchant Electric, (4) Twin Marchant. Two pages are devoted to a listing and discussion of published tables, and some other literature.

13[Z].—L. J. COMRIE, "Recent progress in scientific computing," *J. Sci. Instruments*, v. 21, Aug. 1944, p. 129–135, illustrated.

The substance of this paper was delivered 3 July 1943 at a joint meeting of the London and Home Counties' Branch of the Institute of Physics, and the London Mathematical Society. The headings in the paper are as follows: "Differential analysers," "The training of computers," "Numerical integration by hand," "Finite differences," "Direct and inverse interpolation," "Double-entry interpolation," "Punched-card machines," "Solution of simultaneous equations," "Mathematical tables," "Short bibliography."

NOTES

26. AGETON'S METHOD.—Navigators and other persons who use Hydrographic Office, *Publication*, no. 211 (see *MTAC*, p. 80–81) regularly, will be interested in two notes recently published on Ageton's method of celestial navigation. The first is "The accuracy of Ageton's method in celestial navigation" by SAMUEL HERRICK, Astron. So. Pacific, *Publications*, v. 56, 1944, p. 149–155. Herrick points out that the warning appearing in H.O. 211 against the use of sights for which the value of K is found to lie between the limits of 87° 30′ and 92° 30′ is not adequate. He shows that, when the tables are used in the conventional fashion and K lies outside the forbidden range, the maximum error in the computed altitude, h, is six minutes of arc. He shows also that an error of a minute of arc in h may occur even when K is quite small. He points out that the "caution" mentioned above is unfortunately omitted from Ageton's *Manual of Celestial Navigation*, New York,

Van Nostrand, 1942, T. II, and from three texts which reproduce the tables. Herrick concludes that if a maximum error of two minutes of arc in h is acceptable, Ageton's method can be used without interpolation in the tables so long as K lies outside the range, 82° to 98°.

The second note is "Increased accuracy with Ageton's method" by C. H. SMILEY, *Pop. Astr.*, v. 52, 1944, p. 379–383. Smiley points out that Ageton's method could be used in many of the problems of practical astronomy if its accuracy were increased slightly. He notes that seven-place logarithms of sines and cosines can be used in place of Ageton's A's and B's, otherwise keeping the computing form and rules as given by Ageton. He points out that the indeterminacy in the computed altitude, h, resulting from the value of K being near 90° can often be avoided by interchanging the values of the declination and the latitude. There is a brief discussion of the probability that K will lie near 90°.

C. H. SMILEY

Brown University

27. EARLIEST TABLES OF S AND T.—In N22 A. Fletcher refers to early tables of the auxiliary functions S and T for arguments 0°.01(0°.01)2°.99 given by John Newton in 1658 in his *Trigonometria Britanica*, and quotes a suggestion that these may be the first published tables. Tables embodying an equivalent idea, although only for arguments 0(0°.01)0°.5, were, however, given in the previous year by William Oughtred in the second part of his *Trigonometria*,[1] 1657, which has a separate title-page *Canones Sinuum, Tangentium, Secantium: et Logarithmorum pro Sinubus et Tangentibus*, London, 1657. On p. 235 Oughtred gives quantities which he labels "S" and "A" (being respectively negative and positive); these quantities, given in units of the seventh decimal, are respectively $S - S_0$ and $T - T_0$, in which S_0 and T_0 are the limiting values of S and T for zero argument. The end-figure is not completely reliable (although never more than a unit in error), so that it is not easy to decide whether (i) S and S_0, T and T_0 were individually rounded-off before subtraction in pairs, or whether (ii) the subtractions were first performed with more decimals and the results rounded-off. The former seems more likely since, if we assume this to have been done, Oughtred has 16 errors of $+1$ (i.e. errors in which his "S" is numerically too small and his "T" too large) and 7 of -1; with process (ii), he would have 5 errors of $+1$ and 23 of -1.

Process (i) seems to imply that Oughtred regarded S and T as the primary functions and performed the subtraction simply as a device for giving small numbers to tabulate and interpolate; it is the correct process if the quantities are always to be used in conjunction with the same S_0 and T_0 (i.e. for a specified unit) since the resulting value of S or T is then subject to a slightly smaller total rounding-off error. It seems improbable that Oughtred envisaged any other unit than that of 0°.01 which he used, but it is worth while to note that his "S" and "A," if rounded-off by process (ii) above, would be independent of the unit used, except insofar as this determines the tabular arguments.

J. C. P. MILLER

[1] The writer has seen three copies of this work, not all in agreement in all details. One of these belonged to Isaac Newton, whose library was recently purchased by the Pilgrim Trust and presented to the Library of Trinity College, Cambridge.

28. HENRY GOODWYN. *A Tabular Series . . .* (see *MTAC*, p. 22).—It is perhaps not generally realized that only a part of this table was ever published (one fifth). The complete title is: *A/Tabular Series/of/Decimal Quotients/for all the/Proper Vulgar Fractions/of which/when in their lowest terms/Neither the Numerator nor the Demoninator/is greater than*/1000/London/Printed by W. Marchant, Ingram Court, Fenchurch Street/Published by J. M. Richardson, 23 Cornhill/opposite the Royal Exchange/1823. 15 × 23.5 cm. 5 p. (introduction) + 107 p. (tables) + 5 unpaged leaves.

In his preface Goodwyn writes that he is now publishing Part I of the complete tabular series, consisting of all decimals beginning with .0. Part II is to contain all beginning with .1, and so on to part V containing all decimals beginning with .4. He explains that it is unnecessary to print the remaining decimals from .5 to 1. since they can readily be found as the arithmetical complements of those already tabulated (because in effect $n/d = 1-(d-n)/d$). Thus . . . "the number of entire quotients will amount altogether to 152096, and there being an equal number of complementary (or easily calculated but not printed ones) the total will exhibit the vulgar fractions corresponding to 304192 decimal quotients." He goes on to say that he is now printing Part I, containing all decimals beginning with 0.0, of the *Tabular Series* derived from the values given in his *Table of Circles*, . . . "but whether that table [i.e. of circles] shall be printed, or the present work completed, will depend altogether on the reception which this result of the Computer's labour may meet with from the public."

Then follow the tables from 1/1000 = .001 to 99/991 = .09989909, after which is printed "END OF PART I." Then follow 5 unpaged leaves with 2^n, $n = 0(1)19$; 5^n, $n = 0(1)8$; 2^n5^m, $n = 0(1)6$, $m = 0(1)6$.

It appears therefore that Part I of the *Tabular Series* was first printed, and then (also in 1823) the *Table of Circles*. It is to be presumed that, like most computers of tables, he found that sales did not cover the cost of printing, and no more of the *Tabular Series* was ever printed.

In Glaisher's articles on Goodwyn (see *MTAC*, p. 23) the incomplete state is noted. I imagine that Glaisher had a copy, but it is not in the "Glaisher Collection" in the Cambridge University Library, possibly because the Library already possesses two copies. As Goodwyn's daughter presented her father's 'complete works' in 1831 (see *MTAC*, p. 100) and no subsequent parts accompanied the gift, it may safely be assumed that they do not exist.

C. R. COSENS

QUERIES

11. CUBE ROOTS.—We have recently needed a table of cube roots of numbers $N = .1(.0001).2$, to 6 or 7D. Is such a table available in print?

R. F. BOYER

Dow Chemical Co.,
Midland, Michigan

12. ITEMS IN INTERPOLATION HISTORY.—Who originated (a) The use of the 'throw-back' to take account of higher differences neglected in interpolation, and (b) The use of interpolation formulae involving only even central differences, not advancing differences? Both these methods are generally

supposed to have come into use much more recently than 1826, but they are illustrated by Legendre in volume 2 of his *Traité des Fonctions Elliptiques*, Paris, 1826, paragraphs 672, 673, 676, as the following extracts show:

672, p. 28. "Je remarquerai que lorsque les différences quatrièmes $\delta^4 A$ sont assez grandes pour que les différences suivantes $\delta^5 A$ aient quelqu'influence dans les interpolations, il conviendra de prendre $\delta^4 A - \frac{7}{10}\delta^5 A$ au lieu de $\delta^4 A$." He states that, x being the fraction of tabular interval required in the interpolate (he is here using forward differences) the relevant terms are $\frac{x(x-1)(x-2)(x-3)}{1.2.3.4}\left(\delta^4 A + \frac{x-4}{5}\delta^5 A\right)$; then—"comme $\delta^5 A$ est censé très petit par rapport à $\delta^4 A$, si l'on donne à x une valeur moyenne $\frac{1}{2}$, le terme $\frac{x-4}{5}\delta^5 A$ se reduira à $-\frac{7}{10}\delta^5 A$."

673, p. 29. Similarly a sixth difference can be "thrown back," as we should now say, to the preceding fifth, by taking $\delta^5 A - \frac{3}{4}\delta^6 A$ in place of $\delta^5 A$, $-\frac{3}{4}$ being, as before, the mean value of $\frac{x-5}{6}$ obtained for $x = \frac{1}{2}$.

676, p. 31. "Pour avoir le milieu entre deux termes consécutifs A, A_1 d'une suite dont les différences deviennent progressivement plus petites qu'une quantité donnée, il est bon d'avoir recours aux termes qui précèdent et qui suivent les deux termes proposés."

Legendre is interpolating into the middle of an interval, and in modern notation the formula he gives is the following:

$$f_{\frac{1}{2}} = \frac{f_0 + f_1}{2} - \frac{1}{2}\frac{(\Delta_0^2 + \Delta_1^2)}{8} + \frac{1.3}{2.4}\frac{\Delta_0^4 + \Delta_1^4}{32} - \frac{1.3.5}{2.4.6}\frac{\Delta_0^6 + \Delta_1^6}{128} + \cdots$$

precisely the Everett formula for interpolation at 0.5 tabular interval.

C. R. COSENS

QUERIES—REPLIES

12. LOG LOG TABLES (Q **4**, p. 131; QR **9**, p. 336).—In answer to this question it would seem to be in order to draw attention to Slide Rules with log log scales, which permit ready solution of complicated problems in raising of powers, and extracting a root besides obtaining values of natural logarithms, and hyperbolic functions. For example, to evaluate

$2.31^{5.67}$; $(.371)^{1/15.8}$; $2.87^x = 73.7$; $x^{.373} = .582$; $e^x = .748$; $\ln 31$, $e^{\pi i}$

and various hyperbolic functions of 3.

The Keuffel & Esser Co., Hoboken, N. J., has manufactured (1) The log log duplex trig slide rule with trigonometric scales to represent degrees and minutes; (2) The log log duplex decitrig slide rule, with trigonometric scales to represent degrees and decimals of a degree; (3) The log log duplex vector rule. In Jan. 1943 these rules, 10 in. size, were respectively listed at $12.50, $12.50, $13.50; and the 20 in. size at $27.00, $27.00, and $31.50. Reproductions of the scales of these rules are given in *K & E, Slide Rules and Calculating Instruments*, New York, 1941, p. 313f–313i; and also in figs. 5–7 of the last of the following works:

C. N. PICKWORTH, *The Slide Rule: A Practical Manual*, fourteenth ed.,

Manchester, London, etc., 1916; "Slide rules with log.-log. scales," p. 84–92, "log-log duplex slide rule," p. 115.

J. E. THOMPSON, *A Manual of the Slide Rule, its History, Principle and Operation*, New York, Van Nostrand, 1930; fifth printing, 1942; "The log-log scale and the log-log duplex slide rule," p. 117–133; "Settings and typical problems involving the log-log scales," p. 185–203. Thompson's work is also in T. O. SLOANE, *Speed and Fun with Figures*, New York, Van Nostrand, 1939.

H. O. COOPER, *Slide Rule Calculations*, Oxford Univ. Press, 1931; "Log-log scales," p. 81–91, 118–125.

L. M. KELLS, W. F. KERN & J. R. BLAND, *The Log Log Duplex Decitrig Slide Rule no. 4081, A Manual*, New York, Keuffel & Esser, c. 1943, 103 p.

R. W. FRENCH, *Engineers' Slide Rule*, St. Louis, Mo., J. S. Swift, 1941; "Log log scales," p. 66–83.

J. J. CLARK, *The Slide Rule and Logarithmic Tables including a ten-place table of logarithms . . .*, third ed. *including a table of natural trigonometric functions*, Chicago, F. J. Drake, 1943 [c. 1941], "The log-log scale," p. 90–103.

E. J. HILLS, *A Course in the Slide Rule and Logarithms*, Boston, Ginn, 1943; "The log log scales," p. 84–87.

The first log log slide rules (linear and circular) were invented in 1814 by PETER MARK ROGET (1779–1869) a physician in Manchester and London, and his paper on "Description of a new instrument for performing mechanically the involution and evolution of numbers," R. So. London, *Trans.*, v. 105, 1815, p. 9–28 + plates II–IV, led to his election, in 1815, as a fellow of the Royal Society. He was also a discoverer of the path for the knight moving on a chess-board so as to cover every square once and only once (*Phil. Mag.*, v. 16, 1840, p. 306f, 498f). For a biography of Roget see R. So. London, *Proc.*, v. 18, 1870, p. xxviii–xl. He was secretary of the Society for 21 years.

After Roget there were many others connected with various log log slide rules. For example J. A. BURDON (Acad. d. Sci., Paris, *Comptes Rendus*, v. 58, 1864, p. 573–576), F. BLANC (W. v. DYCK, Deutsche Math.-Ver., *Katalog mathem. u. mathem.-phys. Modelle, Apparate u. Instrumente*, Munich, 1892, p. 145f) from which, in particular, values of sinh x and cosh x could be immediately read off, and W. SCHWETH, Ver. deutscher Ingenieure, *Z.*, v. 45, 1901, p. 567–8, 720. In his article on "Calculating machines" in v. 4, 1929, p. 553, of the *Encyclopædia Britannica*, fourteenth ed., the late D. BAXANDALL stated that "The log-log scale was reinvented and applied to the slide rule by Captain J. H. THOMSON in 1881, and by Prof. JOHN PERRY in 1902." That this statement is incorrect so far as Perry is concerned, at least, may be noted by consulting *Nature*, v. 67, 1902, p. 141, which tells us that Perry was merely making Roget's ideas more generally known. Nevertheless the "Peter and Perry" log log rule was manufactured by Albert Nestler of Lahr, Baden, Germany, and "Perry's new slide rule" by A. G. Thornton of Manchester, England (see *Engineering Record*, v. 58, Dec. 12, 1908, current news suppl., p. 37, and Pickworth's *Slide Rule*, p. 91f). Pickworth describes also log log rules introduced by John Davis & Son, Ltd., of Derby, England (H. C. DUNLOP and C. S. JACKSON, *Slide Rule Notes*, London, 1901); and A. W. Faber, of Stein (near Nürnberg), Germany. Thomson's rule was exhibited in the International Exhibition of Inventions, London, 1885, with full recognition of its earlier discovery by Roget (*Van Nostrand's Engin.*

Mag., v. 33, 1885, p. 517). More information about log log slide rules is contained in Baxandall's *Catalogue of the Collections in the Science Museum, South Kensington . . . Mathematics I. Calculating Machines and Instruments*, London, His Majesty's Stationery Office, 1926, p. 56–58.

R. C. A.

13. TABLES OF $N^{3/2}$ (Q **5**, p. 131; QR **8**, p. 204, **11**, p. 336).—A further contribution to the bibliography of these tables is *American Civil Engineers' Handbook*, ed. by T. MERRIMAN and T. H. WIGGIN, fifth ed., New York, Wiley, 1930, p. 1312–1314. This table is for $N = [.01(.01).1(.1)10(.5)20; 4S]$.

In *MTAC*, p. 336 a reference was given to T. 38 in King's *Handbook of Hydraulics*, third ed. 1939, p. 103–112. In this same work, p. 117–121, is T. 40 "Discharge in cubic feet per second per foot of length over sharp-crested wiers, without velocity of approach correction by the Francis formula $Q = 3.33H^{3/2}$," for $H = [.0(.001)1.5(.01)6.99; 5S]$, the values Q of T. 40 being 3.33 times corresponding values in T. 38. A slide rule giving the values of the Francis' wier formula, $Q = 3.33H^{3/2}(B - .2H)$ is illustrated in J. N. ARNOLD, *Special Slide Rules*, Purdue Univ., Lafayette, Indiana, *Engineering Bulletin*, v. 17, no. 5, 1933, p. 14–15. In this formula Q is, in cu. ft./sec., the water discharge, with head of water H, of a rectangular notch wier of breadth B ft. Furthermore, the values of $N^{3/2}$, to a certain accuracy, could be read off at once from log log slide rules discussed in QR **12**.

R. C. A.

CORRIGENDA ET ADDENDA

P. 211, l. 20, *omit* of the first kind.

P. 215, l. 17, end of line, *for* ($p = 2, 4, 6$), *read* ($p = 4, 6, 8$).

P. 217, C_2 6, *add* , $p = 1(1)18$.

P. 220, last l., *for* δ^2, *read* δ_m^2. P. 225, l. 12, *for* $w_{-n}(-x)$, *read* $w_n(-x)$.

P. 231, l. −11, *for* odd integer, *read* odd-integer.

P. 234, D8A, l. 5, *for* a_n, *read* a_2

P. 240, l. −3, in two places, *for* $x^{\frac{1}{2}}$, *read* x^2; l. −5, *for* 10, *read* 8; l. −7, before the integral, *add*: $2x^{n-m+1}e^{-x^2}p^{m-n+1}$.

P. 252, l. 23–24, substitute the following sentence: The terms $\mathrm{her}_n x$, $\mathrm{hei}_n x$ are given in Watson's *Bessel Functions*, p. 81 and are used in Dwight 3_1; $\mathrm{yer}_n x$, $\mathrm{yei}_n x$ were added by J. C. P. MILLER in the Liverpool *Index*.

P. 256, E2, l. 1, *for* $(X/V)^{\frac{1}{2}}$, *read* $\frac{1}{2}x(X/V)^{\frac{1}{2}}$.

P. 257, after entry E14, *add* For improved forms of θ and ϕ given in nos. 1–14 we are indebted to the Liverpool *Index*.

P. 271, for l. 12 *read* The asymptotic forms of the ber, bei, ker, kei, functions and their derivatives are given in the natural form by Dwight 1_2 and 3_1. A modified form quoted by Watson is

P. 272, l. 2, *read* $u = \frac{1}{4}\pi(2r - 1 + 4s)$.

P. 285, l. 32, *for* OSAKA and, *read* OKAYA &.

P. 287, l. 10, *for* DINNIK 15, *read* DINNIK 14.

P. 292, l. 12, *for* f *read* J.

P. 295, l. 19, *for* 1/3, Prescott, *read* 1/3; Prescott.

P. 300, NYMTP 8, l. 1–3, *delete* $K_0(x) = E_0(x)$ . . . $x = 0(.001).03$; and.

P. 330, l. 11, *for* $m = 20$, *read* $n = 20$.

P. 333, l. 16, and 17, editorial slips for which the author was not responsible: *for* 0(0°.001)3°, *read* 0°.01(0°.01)2°.99, and *for* 0°.001, *read* 0°.01.

CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER
Professor KETCHUM, Professor LOWAN

F. Theory of Numbers
Professor LEHMER

G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
J. Summation of Series

I. Finite Differences. Interpolation
K. Statistics
Professor WILKS, *chairman*, Professor COCHRAN, Professor CRAIG
Professor EISENHART, Doctor SHEWHART

L. Higher Mathematical Functions
M. Integrals
Professor BATEMAN

N. Interest and Investment
O. Actuarial Science
Mister ELSTON, *chairman*, Mister THOMPSON, Mister WILLIAMSON

P. Engineering

Q. Astronomy
Doctor ECKERT, *chairman*, Doctor GOLDBERG, Miss KRAMPE

R. Geodesy
S. Physics
T. Chemistry
U. Navigation

Z. Calculating Machines and Mechanical Computation
Doctor COMRIE, *chairman*, Professor CALDWELL, *vice-chairman*
Professor LEHMER, Doctor MILLER, Doctor STIBITZ, Professor TRAVIS

EDITORIAL AND OTHER NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

Since subscriptions to *MTAC* are *payable in advance*, delay in payment for 1945 will cause corresponding delay in mailing Number 9. No sample copies are distributed.

Volume I is to contain 12 numbers continuously paged.

CONTENTS

JANUARY 1945

Machines for Solving Algebraic Equations 337

Zeros of Certain Bessel Functions of Fractional Order 353

Recent Mathematical Tables

Review of Tables by **169** (BAASMTC, NYMTP); **170** (Marchant Calculating Machine Co.); **171** (Linnebach); **172** (National Defense Research Committee); **173** (Pucher); **174** (Wright); **175** (Lowan & Salzer); **176** (Salzer); **177** (Stenzel); **178** (Bose); **179** (BAASMTC); **180** (Federal Telephone and Radio Corp.); **181** (NYMTP); **182** (Watson); **183** (Lindemann) 355

Mathematical Tables—Errata

55 (Bates); **56** (Forsyth); **57** (Karas); **58** (Lommel, Watson); **59** (NYMTP); **60** (Watson) 364

Unpublished Mathematical Tables

29 (Heppel); **30** (Thomson); **31** (Sang) 367

Mechanical Aids to Computation

12 (Comrie); **13** (Comrie) 370

Notes

26. Ageton's Method; **27.** Earliest Tables of S and T; **28.** Henry Goodwyn, *A Tabular Series* 370

Queries

11. Cube Roots; **12.** Items in Interpolation History 372

Queries—Replies

12. Log Log Tables; **13.** Tables of $N^{3/2}$ 373

Corrigenda et Addenda 375

LANCASTER PRESS, INC., LANCASTER, PA.

337
353
355
364
367
370
370
372
373
375

ER, PA.